Appendix
Mathematics, Symbols, and Physical Constants

© 2003 by CRC Press LLC

Greek Alphabet

Greek Letter			Greek Name	EnglishEquivalent	Greek Letter			Greek Name	English Equivalent
Α	α		Alpha	a	Ν	ν		Nu	n
Β	β		Beta	b	Ξ	ξ		Xi	x
Γ	γ		Gamma	g	Ο	ο		Omicron	o
Δ	δ		Delta	d	Π	π		Pi	p
Ε	ε		Epsilon	e	Ρ	ρ		Rho	r
Ζ	ζ		Zeta	z	Σ	σ	s	Sigma	s
Η	η		Eta	ē	Τ	τ		Tau	t
Θ	θ	ϑ	Theta	th	Υ	υ		Upsilon	u
Ι	ι		Iota	i	Φ	ϕ	φ	Phi	ph
Κ	κ		Kappa	k	Χ	χ		Chi	ch
Λ	λ		Lambda	l	Ψ	ψ		Psi	ps
Μ	μ		Mu	m	Ω	ω		Omega	ō

© 2003 by CRC Press LLC

International System of Units (SI)

The International System of Units (SI) was adopted by the 11th General Conference on Weights and Measures (CGPM) in 1960. It is a coherent system of units built from seven *SI base units,* one for each of the seven dimensionally independent base quantities: the meter, kilogram, second, ampere, kelvin, mole, and candela, for the dimensions length, mass, time, electric current, thermodynamic temperature, amount of substance, and luminous intensity, respectively. The definitions of the SI base units are given below. The *SI derived units* are expressed as products of powers of the base units, analogous to the corresponding relations between physical quantities but with numerical factors equal to unity.

In the International System there is only one SI unit for each physical quantity. This is either the appropriate SI base unit itself or the appropriate SI derived unit. However, any of the approved decimal prefixes, called *SI prefixes,* may be used to construct decimal multiples or submultiples of SI units.

It is recommended that only SI units be used in science and technology (with SI prefixes where appropriate). Where there are special reasons for making an exception to this rule, it is recommended always to define the units used in terms of SI units. This section is based on information supplied by IUPAC.

Definitions of SI Base Units

Meter — The meter is the length of path traveled by light in vacuum during a time interval of 1/299 792 458 of a second (17th CGPM, 1983).

Kilogram — The kilogram is the unit of mass; it is equal to the mass of the international prototype of the kilogram (3rd CGPM, 1901).

Second — The second is the duration of 9 192 631 770 periods of the radiation corresponding to the transition between the two hyperfine levels of the ground state of the cesium-133 atom (13th CGPM, 1967).

Ampere — The ampere is that constant current which, if maintained in two straight parallel conductors of infinite length, of negligible circular cross-section, and placed 1 meter apart in vacuum, would produce between these conductors a force equal to 2×10^{-7} newton per meter of length (9th CGPM, 1948).

Kelvin — The kelvin, unit of thermodynamic temperature, is the fraction 1/273.16 of the thermodynamic temperature of the triple point of water (13th CGPM, 1967).

Mole — The mole is the amount of substance of a system that contains as many elementary entities as there are atoms in 0.012 kilogram of carbon-12. When the mole is used, the elementary entities must be specified and may be atoms, molecules, ions, electrons, or other particles, or specified groups of such particles (14th CGPM, 1971).

Examples of the use of the mole:

1 mol of H_2 contains about 6.022×10^{23} H_2 molecules, or 12.044×10^{23} H atoms
1 mol of HgCl has a mass of 236.04 g
1 mol of Hg_2Cl_2 has a mass of 472.08 g
1 mol of Hg_2^{2+} has a mass of 401.18 g and a charge of 192.97 kC
1 mol of $Fe_{0.91}S$ has a mass of 82.88 g
1 mol of e^- has a mass of 548.60 μg and a charge of −96.49 kC
1 mol of photons whose frequency is 10^{14} Hz has energy of about 39.90 kJ

Candela — The candela is the luminous intensity, in a given direction, of a source that emits monochromatic radiation of frequency 540×10^{12} hertz and that has a radiant intensity in that direction of (1/683) watt per steradian (16th CGPM, 1979).

© 2003 by CRC Press LLC

Names and Symbols for the SI Base Units

Physical Quantity	Name of SI Unit	Symbol for SI Unit
Length	Meter	m
Mass	Kilogram	kg
Time	Second	s
Electric current	Ampere	A
Thermodynamic temperature	Kelvin	K
Amount of substance	Mole	mol
Luminous intensity	Candela	cd

SI Derived Units with Special Names and Symbols

Physical Quantity	Name of SI Unit	Symbol for SI Unit	Expression in Terms of SI Base Units
Frequency [1]	Hertz	Hz	s^{-1}
Force	Newton	N	$m\ kg\ s^{-2}$
Pressure, stress	Pascal	Pa	$N\ m^{-2} = m^{-1}\ kg\ s^{-2}$
Energy, work, heat	Joule	J	$N\ m = m^2\ kg\ s^{-2}$
Power, radiant flux	Watt	W	$J\ s^{-1} = m^2\ kg\ s^{-3}$
Electric charge	Coulomb	C	A s
Electric potential, electromotive force	Volt	V	$J\ C^{-1} = m^2\ kg\ s^{-3} A^{-1}$
Electric resistance	Ohm	Ω	$V\ A^{-1} = m^2\ kg\ s^{-3} A^{-2}$
Electric conductance	Siemens	S	$\Omega^{-1} = m^{-2}\ kg^{-1}\ s^3 A^2$
Electric capacitance	Farad	F	$C\ V^{-1} = m^{-2}\ kg^{-1}\ s^4 A^2$
Magnetic flux density	Tesla	T	$V\ s\ m^{-2} = kg\ s^{-2} A^{-1}$
Magnetic flux	Weber	Wb	$V\ s = m^2\ kg\ s^{-2} A^{-1}$
Inductance	Henry	H	$V\ A^{-1}\ s = m^2\ kg\ s^{-2} A^{-2}$
Celsius temperature [2]	Degree Celsius	°C	K
Luminous flux	Lumen	lm	cd sr
Illuminance	Lux	lx	$cd\ sr\ m^{-2}$
Activity (radioactive)	Becquerel	Bq	s^{-1}
Absorbed dose (of radiation)	Gray	Gy	$J\ kg^{-1} = m^2\ s^{-2}$
Dose equivalent (dose equivalent index)	Sievert	Sv	$J\ kg^{-1} = m^2\ s^{-2}$
Plane angle	Radian	rad	$I = m\ m^{-1}$
Solid angle	Steradian	sr	$I = m^2\ m^{-2}$

[1] For radial (circular) frequency and for angular velocity, the unit rad s^{-1}, or simply s^{-1}, should be used, and this may not be simplified to Hz. The unit Hz should be used only for frequency in the sense of cycles per second.

[2] The Celsius temperature θ is defined by the equation:

$$\theta/°C = T/K - 273.15$$

The SI unit of Celsius temperature interval is the degree Celsius, °C, which is equal to the kelvin, K. °C should be treated as a single symbol, with no space between the ° sign and the letter C. (The symbol °K, and the symbol °, should no longer be used.)

© 2003 by CRC Press LLC

Units in Use Together with the SI

These units are not part of the SI, but it is recognized that they will continue to be used in appropriate contexts. SI prefixes may be attached to some of these units, such as milliliter, ml; millibar, mbar; megaelectronvolt, MeV; and kilotonne, ktonne.

Physical Quantity	Name of Unit	Symbol for Unit	Value in SI Units
Time	Minute	min	60 s
Time	Hour	h	3600 s
Time	Day	d	86 400 s
Planeangle	Degree	°	$(\pi/180)$ rad
Planeangle	Minute	′	$(\pi/10\ 800)$ rad
Planeangle	Second	″	$(\pi/648\ 000)$ rad
Length	Ångstrom [1]	Å	10^{-10} m
Area	Barn	b	10^{-28} m^2
Volume	Liter	l, L	dm^3 = 10^{-3} m^3
Mass	Tonne	t	Mg = 10^3 kg
Pressure	Bar [1]	bar	10^5 Pa = 10^5 N m^{-2}
Energy	Electronvolt [2]	eV (= $e \times$ V)	$\approx 1.60218 \times 10^{-19}$ J
Mass	Unified atomic mass unit [2,3]	u (= $m_a(^{12}C)/12$)	$\approx 1.66054 \times 10^{-27}$ kg

[1] The ångstrom and the bar are approved by CIPM for "temporary use with SI units," until CIPM makes a further recommendation. However, they should not be introduced where they are not used at present.
[2] The values of these units in terms of the corresponding SI units are not exact, since they depend on the values of the physical constants e (for the electronvolt) and N_A (for the unified atomic mass unit), which are determined by experiment.
[3] The unified atomic mass unit is also sometimes called the dalton, with symbol Da, although the name and symbol have not been approved by CGPM.

Conversion Constants and Multipliers

Recommended Decimal Multiples and Submultiples

Multiples and Submultiples	Prefixes	Symbols	Multiples and Submultiples	Prefixes	Symbols
10^{18}	exa	E	10^{-1}	deci	d
10^{15}	peta	P	10^{-2}	centi	c
10^{12}	tera	T	10^{-3}	milli	m
10^9	giga	G	10^{-6}	micro	μ (Greek mu)
10^6	mega	M	10^{-9}	nano	n
10^3	kilo	k	10^{-12}	pico	p
10^2	hecto	h	10^{-15}	femto	f
10	deca	da	10^{-18}	atto	a

© 2003 by CRC Press LLC

Conversion Factors — Metric to English

To obtain	Multiply	By
Inches	Centimeters	0.3937007874
Feet	Meters	3.280839895
Yards	Meters	1.093613298
Miles	Kilometers	0.6213711922
Ounces	Grams	$3.527396195 \times 10^{-2}$
Pounds	Kilograms	2.204622622
Gallons (U.S. liquid)	Liters	0.2641720524
Fluid ounces	Milliliters (cc)	$3.381402270 \times 10^{-2}$
Square inches	Square centimeters	0.1550003100
Square feet	Square meters	10.76391042
Square yards	Square meters	1.195990046
Cubic inches	Milliliters (cc)	$6.102374409 \times 10^{-2}$
Cubic feet	Cubic meters	35.31466672
Cubic yards	Cubic meters	1.307950619

Conversion Factors — English to Metric*

To obtain	Multiply	By
Microns	Mils	**25.4**
Centimeters	Inches	**2.54**
Meters	Feet	**0.3048**
Meters	Yards	**0.9144**
Kilometers	Miles	**1.609344**
Grams	Ounces	28.34952313
Kilograms	Pounds	**0.45359237**
Liters	Gallons (U.S. liquid)	**3.785411784**
Millimeters (cc)	Fluid ounces	29.57352956
Square centimeters	Square inches	**6.4516**
Square meters	Square feet	**0.09290304**
Square meters	Square yards	**0.83612736**
Milliliters (cc)	Cubic inches	**16.387064**
Cubic meters	Cubic feet	$2.831684659 \times 10^{-2}$
Cubic meters	Cubic yards	0.764554858

* Boldface numbers are exact; others are given to ten significant figures where so indicated by the multiplier factor.

Conversion Factors — General*

To obtain	Multiply	By
Atmospheres	Feet of water @ 4°C	2.950×10^{-2}
Atmospheres	Inches of mercury @ 0°C	3.342×10^{-2}
Atmospheres	Pounds per square inch	6.804×10^{-2}
BTU	Foot-pounds	1.285×10^{-3}
BTU	Joules	9.480×10^{-4}
Cubic feet	Cords	**128**
Degree (angle)	Radians	57.2958
Ergs	Foot-pounds	1.356×10^{7}
Feet	Miles	**5280**
Feet of water @ 4°C	Atmospheres	33.90
Foot-pounds	Horsepower-hours	1.98×10^{6}
Foot-pounds	Kilowatt-hours	2.655×10^{6}
Foot-pounds per min	Horsepower	3.3×10^{4}
Horsepower	Foot-pounds per sec	1.818×10^{-3}
Inches of mercury @ 0°C	Pounds per square inch	2.036

© 2003 by CRC Press LLC

To obtain	Multiply	By
Joules	BTU	1054.8
Joules	Foot-pounds	1.35582
Kilowatts	BTU per min	1.758×10^{-2}
Kilowatts	Foot-pounds per min	2.26×10^{-5}
Kilowatts	Horsepower	0.745712
Knots	Miles per hour	0.86897624
Miles	Feet	1.894×10^{-4}
Nautical miles	Miles	0.86897624
Radians	Degrees	1.745×10^{-2}
Square feet	Acres	**43560**
Watts	BTU per min	17.5796

* Boldface numbers are exact; others are given to ten significant figures where so indicated by the multiplier factor.

Temperature Factors

$$°F = 9/5(°C) + 32$$

$$\text{Fahrenheit temperature} = 1.8\ (\text{temperature in kelvins}) - 459.67$$

$$°C = 5/9[(°F) - 32]$$

$$\text{Celsius temperature} = \text{temperature in kelvins} - 273.15$$
$$\text{Fahrenheit temperature} = 1.8\ (\text{Celsius temperature}) + 32$$

Conversion of Temperatures

From	To	
°Celsius	°Fahrenheit	$t_F = (t_C \times 1.8) + 32$
	Kelvin	$T_K = t_C + 273.15$
	°Rankine	$T_R = (t_C + 273.15) \times 18$
°Fahrenheit	°Celsius	$t_C = \frac{t_F - 32}{1.8}$
	Kelvin	$T_K = \frac{t_F - 32}{1.8} + 273.15$
	°Rankine	$T_R = t_F + 459.67$
Kelvin	°Celsius	$t_C = T_K - 273.15$
	°Rankine	$T_R = T_K \times 1.8$
°Rankine	°Fahrenheit	$t_F = T_R - 459.67$
	Kelvin	$T_K = \frac{T_R}{1.8}$

Physical Constants

General

Equatorial radius of the earth = 6378.388 km = 3963.34 miles (statute).
Polar radius of the earth = 6356.912 km = 3949.99 miles (statute).
1 degree of latitude at 40° = 69 miles.

© 2003 by CRC Press LLC

1 international nautical mile = 1.15078 miles (statute) = 1852 m = 6076.115 ft.
Mean density of the earth = 5.522 g/cm^3 = 344.7 lb/ft^3.
Constant of gravitation $(6.673 \pm 0.003) \times 10^{-8}$ cm^3 gm^{-1}s^{-2}.
Acceleration due to gravity at sea level, latitude 45° = 980.6194 cm/s^2 = 32.1726 ft/s^2.
Length of seconds pendulum at sea level, latitude 45° = 99.3575 cm = 39.1171 in.
1 knot (international) = 101.269 ft/min = 1.6878 ft/s = 1.1508 miles (statute)/h.
1 micron = 10^{-4} cm.
1 ångstrom = 10^{-8} cm.
Mass of hydrogen atom = $(1.67339 \pm 0.0031) \times 10^{-24}$ g.
Density of mercury at 0°C = 13.5955 g/ml.
Density of water at 3.98°C = 1.000000 g/ml.
Density, maximum, of water, at 3.98°C = 0.999973 g/cm^3.
Density of dry air at 0°C, 760 mm = 1.2929 g/l.
Velocity of sound in dry air at 0°C = 331.36 m/s = 1087.1 ft/s.
Velocity of light in vacuum = $(2.997925 \pm 0.000002) \times 10^{10}$ cm/s.
Heat of fusion of water 0°C = 79.71 cal/g.
Heat of vaporization of water 100°C = 539.55 cal/g.
Electrochemical equivalent of silver = 0.001118 g/s international amp.
Absolute wavelength of red cadmium light in air at 15°C, 760 mm pressure = 6438.4696 Å.
Wavelength of orange-red line of krypton 86 = 6057.802 Å.

π Constants

π = 3.14159 26535 89793 23846 26433 83279 50288 41971 69399 37511
$1/\pi$ = 0.31830 98861 83790 67153 77675 26745 02872 40689 19291 48091
π^2 = 9.8690 44010 89358 61883 44909 99876 15113 53136 99407 24079
$\log_e \pi$ = 1.14472 98858 49400 17414 34273 51353 05871 16472 94812 91531
$\log_{10} \pi$ = 0.49714 98726 94133 85435 12682 88290 89887 36516 78324 38044
$\mathrm{og}_{10}\sqrt{2\pi}$ = 0.39908 99341 79057 52478 25035 91507 69595 02099 34102 92128

Constants Involving *e*

e = 2.71828 18284 59045 23536 02874 71352 66249 77572 47093 69996
$1/e$ = 0.36787 94411 71442 32159 55237 70161 46086 74458 11131 03177
e^2 = 7.38905 60989 30650 22723 04274 60575 00781 31803 15570 55185
$M = \log_{10} e$ = 0.43429 44819 03251 82765 11289 18916 60508 22943 97005 80367
$1/M = \log_e 10$ = 2.30258 50929 94045 68401 79914 54684 36420 76011 01488 62877
$\log_{10} M$ = 9.63778 43113 00536 78912 29674 98645 – 10

Numerical Constants

$\sqrt{2}$ = 1.41421 35623 73095 04880 16887 24209 69807 85696 71875 37695
$\sqrt[3]{2}$ = 1.25992 10498 94873 16476 72106 07278 22835 05702 51464 70151
$\log_e 2$ = 0.69314 71805 59945 30941 72321 21458 17656 80755 00134 36026
$\mathrm{og}_{10} 2$ = 0.30102 99956 63981 19521 37388 94724 49302 67881 89881 46211
$\sqrt{3}$ = 1.73205 08075 68877 29352 74463 41505 87236 69428 05253 81039
$\sqrt[3]{3}$ = 1.44224 95703 07408 38232 16383 10780 10958 83918 69253 49935
$\log_e 3$ = 1.09861 22886 68109 69139 52452 36922 52570 46474 90557 82275
$\mathrm{og}_{10} 3$ = 0.47712 12547 19662 43729 50279 03255 11530 92001 28864 19070

© 2003 by CRC Press LLC

Symbols and Terminology for Physical and Chemical Quantities

Name	Symbol	Definition	SI unit
		Classical Mechanics	
Mass	m		kg
Reduced mass	μ	$\mu = m_1 m_2/(m_1 + m_2)$	kg
Density, mass density	ρ	$\rho = m/V$	kg m^{-3}
Relative density	d	$d = \rho/\rho^\theta$	1
Surface density	ρ_A, ρ_S	$\rho_A = m/A$	kg m^{-2}
Specific volume	v	$v = V/m = 1/\rho$	m^3 kg^{-1}
Momentum	p	$\boldsymbol{p} = mv$	kg m s^{-1}
Angular momentum, action	L	$\boldsymbol{L} = r \times p$	J s
Moment of inertia	I, J	$l = \Sigma\, m_i r_i^2$	kg m^2
Force	F	$\boldsymbol{F} = \mathrm{d}\boldsymbol{p}/\mathrm{d}t = m\boldsymbol{a}$	N
Torque, moment of a force	$\boldsymbol{T}$, ($\boldsymbol{M}$)	$\boldsymbol{T} = r \times \boldsymbol{F}$	N m
Energy	E		J
Potential energy	E_p, V, Φ	$E_\mathrm{p} = -\int \boldsymbol{F} \cdot \mathrm{d}s$	J
Kinetic energy	E_k, T, K	$E_\mathrm{k} = (1/2)mv^2$	J
Work	W, w	$W = \int F \cdot \mathrm{d}s$	J
Hamilton function	H	$H(q, p) = T(q, p) + V(q)$	J
Lagrange function	L	$L(q, \dot{q}) = T(q, \dot{q}) - V(q)$	J
Pressure	p, P	$p = F/A$	Pa, N m^{-2}
Surface tension	γ, σ	$\gamma = \mathrm{d}W/\mathrm{d}A$	N m^{-1}, J m^{-2}
Weight	G, (W, P)	$G = mg$	N
Gravitational constant	G	$F = Gm_1 m_2/r^2$	N m^2 kg^{-2}
Normal stress	σ	$\sigma = F/A$	Pa
Shear stress	τ	$\tau = F/A$	Pa
Linear strain, relative elongation	ε, e	$\varepsilon = \Delta l/l$	l
Modulus of elasticity, Young's modulus	E	$E = \sigma/\varepsilon$	Pa
Shear strain	γ	$\gamma = \Delta x/d$	l
Shear modulus	G	$\mathrm{G} = \tau/\gamma$	Pa
Volume strain, bulk strain	θ	$\theta = \Delta V/V_0$	1
Bulk modulus	K	$K = -V_0\,(\mathrm{d}p/\mathrm{d}V)$	Pa
Compression modulus	η, μ	$\tau_{x,z} = \eta(\mathrm{d}v_x/\mathrm{d}z)$	Pa s
Viscosity, dynamic viscosity, fluidity	ϕ	$\phi = 1/\eta$	m kg^{-1} s
Kinematic viscosity	ν	$\nu = \eta/\rho$	m^2 s^{-1}
Friction coefficient	μ, (f)	$F_\mathrm{frict} = \mu F_\mathrm{norm}$	l
Power	P	$P = \mathrm{d}W/\mathrm{d}t$	W
Sound energy flux	P, P_a	$P = \mathrm{d}E/\mathrm{d}t$	W
Acoustic factors			
Reflection factor	ρ	$\rho = P_\mathrm{r}/P_0$	1
Acoustic absorption factor	α_a, (α)	$\alpha_\mathrm{a} = 1 - \rho$	1
Transmission factor	τ	$\tau = P_\mathrm{tr}/P_0$	1
Dissipation factor	δ	$\delta = \alpha_\mathrm{a} - \tau$	1

Elementary Algebra and Geometry

Fundamental Properties (Real Numbers)

$a + b = b + a$	Commutative Law for Addition
$(a + b) + c = a + (b + c)$	Associative Law for Addition

© 2003 by CRC Press LLC

$a + 0 = 0 + a$	Identity Law for Addition
$a + (-a) = (-a) + a = 0$	Inverse Law for Addition
$a(bc) = (ab)c$	Associative Law for Multiplication
$a\left(\frac{1}{a}\right) = \left(\frac{1}{a}\right)a = 1, a \neq 0$	Inverse Law for Multiplication
$(a)(1) = (1)(a) = a$	Identity Law for Multiplication
$ab = ba$	Commutative Law for Multiplication
$a(b + c) = ab + ac$	Distributive Law

DIVISION BY ZERO IS NOT DEFINED

Exponents

For integers m and n

$$a^n a^m = a^{n+m}$$
$$a^n / a^m = a^{n-m}$$
$$(a^n)^m = a^{nm}$$
$$(ab)^m = a^m b^m$$
$$(a/b)^m = a^m / b^m$$

Fractional Exponents

$$a^{p/q} = (a^{1/q})^p$$

where $a^{1/q}$ is the positive qth root of a if $a > 0$ and the negative qth root of a if a is negative and q is odd. Accordingly, the five rules of exponents given above (for integers) are also valid if m and n are fractions, provided a and b are positive.

Irrational Exponents

If an exponent is irrational, e.g., $\sqrt{2}$, the quantity, such as $a^{\sqrt{2}}$, is the limit of the sequence, $a^{1.4}, a^{1.41}, a^{1.414}, \ldots$.

Operations with Zero

$$0^m = 0; \; a^0 = 1$$

Logarithms

If x, y, and b are positive and $b \neq 1$

$$\log_b(xy) = \log_b x + \log_b y$$
$$\log_b(x/y) = \log_b x - \log_b y$$
$$\log_b x^p = p \log_b x$$
$$\log_b(1/x) = -\log_b x$$
$$\log_b b = 1$$
$$\log_b 1 = 0 \qquad \textit{Note: } b^{\log_b x} = x$$

© 2003 by CRC Press LLC

Change of Base ($a \neq 1$)

$$\log_b x = \log_a x \log_b a$$

Factorials

The factorial of a positive integer n is the product of all the positive integers less than or equal to the integer n and is denoted $n!$. Thus,

$$n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n$$

Factorial 0 is defined: $0! = 1$.

Stirling's Approximation

$$\lim_{n \to \infty} (n/e)^n \sqrt{2\pi n} = n!$$

Binomial Theorem

For positive integer n

$$(x+y)^n = x^n + nx^{n-1}y + \frac{n(n-1)}{2!}x^{n-2}y^2 + \frac{n(n-1)(n-2)}{3!}x^{n-3}y^3 + \cdots + nxy^{n-1} + y^n$$

Factors and Expansion

$$\begin{aligned}
(a+b)^2 &= a^2 + 2ab + b^2 \\
(a-b)^2 &= a^2 - 2ab + b^2 \\
(a+b)^3 &= a^3 + 3a^2b + 3ab^2 + b^3 \\
(a-b)^3 &= a^3 - 3a^2b + 3ab^2 - b^3 \\
(a^2 - b^2) &= (a-b)(a+b) \\
(a^3 - b^3) &= (a-b)(a^2 + ab + b^2) \\
(a^3 + b^3) &= (a+b)(a^2 - ab + b^2)
\end{aligned}$$

Progression

An *arithmetic progression* is a sequence in which the difference between any term and the preceding term is a constant (d):

$$a, a+d, a+2d, \ldots, a+(n-1)d$$

If the last term is denoted l [$= a + (n-1)d$], then the sum is

$$s = \frac{n}{2}(a+l)$$

A *geometric progression* is a sequence in which the ratio of any term to the preceding term is a constant r. Thus, for n terms

$$a, ar, ar^2, \ldots, ar^{n-1}$$

© 2003 by CRC Press LLC

the sum is

$$S = \frac{a - ar^n}{1 - r}$$

Complex Numbers

A complex number is an ordered pair of real numbers (a, b).

Equality: $(a, b) = (c, d)$ if and only if $a = c$ and $b = d$
Addition: $(a, b) + (c, d) = (a + c, b + d)$
Multiplication: $(a, b)(c, d) = (ac - bd, ad + bc)$

The first element (a, b) is called the *real* part; the second is the *imaginary* part. An alternate notation for (a, b) is $a + bi$, where $i^2 = (-1, 0)$, and $i = (0, 1)$ or $0 + 1i$ is written for this complex number as a convenience. With this understanding, i behaves as a number, i.e., $(2 - 3i)(4 + i) = 8 - 12i + 2i - 3i^2 = 11 - 10i$. The conjugate of $a + bi$ is $a - bi$ and the product of a complex number and its conjugate is $a^2 + b^2$. Thus, *quotients* are computed by multiplying numerator and denominator by the conjugate of the denominator, as illustrated below:

$$\frac{2 + 3i}{4 + 2i} = \frac{(4 - 2i)(2 + 3i)}{(4 - 2i)(4 + 2i)} = \frac{14 + 8i}{20} = \frac{7 + 4i}{10}$$

Polar Form

The complex number $x + iy$ may be represented by a plane vector with components x and y

$$x + iy = r(\cos\theta + i\sin\theta)$$

(see Figure 1). Then, given two complex numbers $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, the product and quotient are

$$\begin{aligned}
\textbf{Product:} \quad & z_1 z_2 = r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)] \\
\textbf{Quotient:} \quad & z_1/z_2 = (r_1/r_2)[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)] \\
\textbf{Powers:} \quad & z^n = [r(\cos\theta + i\sin\theta)]^n = r^n[\cos n\theta + i\sin n\theta] \\
\textbf{Roots:} \quad & z^{1/n} = [r(\cos\theta + i\sin\theta)]^{1/n} \\
& \quad = r^{1/n}\left[\cos\frac{\theta + k.360}{n} + i\sin\frac{\theta + k.360}{n}\right], \qquad k = 0, 1, 2, \ldots, n - 1
\end{aligned}$$

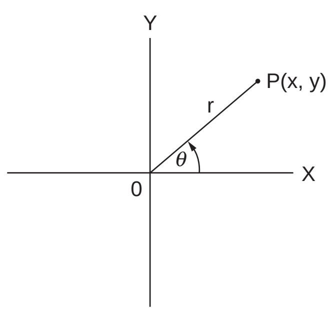


FIGURE 1 Polar form of complex number.

© 2003 by CRC Press LLC

Permutations

A permutation is an ordered arrangement (sequence) of all or part of a set of objects. The number of permutations of n objects taken r at a time is

$$\begin{aligned} p(n, r) &= n(n-1)(n-2)\ldots(n-r+1) \\ &= \frac{n!}{(n-r)!} \end{aligned}$$

A permutation of positive integers is "even" or "odd" if the total number of inversions is an even integer or an odd integer, respectively. Inversions are counted relative to each integer j in the permutation by counting the number of integers that follow j and are less than j. These are summed to give the total number of inversions. For example, the permutation 4132 has four inversions: three relative to 4 and one relative to 3. This permutation is therefore even.

Combinations

A combination is a selection of one or more objects from among a set of objects regardless of order. The number of combinations of n different objects taken r at a time is

$$C(n, r) = \frac{P(n, r)}{r!} = \frac{n!}{r!(n-r)!}$$

Algebraic Equations

Quadratic

If $ax^2 + bx + c = 0$, and $a \neq 0$, then roots are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Cubic

To solve $x^3 + bx^2 + cx + d = 0$, let $x = y - b/3$. Then the *reduced cubic* is obtained:

$$y^3 + py + q = 0$$

where $p = c - (1/3)b^2$ and $q = d - (1/3)bc + (2/27)b^3$. Solutions of the original cubic are then in terms of the reduced cubic roots y_1, y_2, y_3:

$$x_1 = y_1 - (1/3)b \qquad x_2 = y_2 - (1/3)b \qquad x_3 = y_3 - (1/3)b$$

The three roots of the reduced cubic are

$$\begin{aligned} y_1 &= (A)^{1/3} + (B)^{1/3} \\ y_2 &= W(A)^{1/3} + W^2(B)^{1/3} \\ y_3 &= W^2(A)^{1/3} + W(B)^{1/3} \end{aligned}$$

where

$$A = -\frac{1}{2}q + \sqrt{(1/27)p^3 + \frac{1}{4}q^2}$$

© 2003 by CRC Press LLC

$$B = -\frac{1}{2}q - \sqrt{(1/27)p^3 + \frac{1}{4}q^2}$$

$$W = \frac{-1 + i\sqrt{3}}{2}, \quad W^2 = \frac{-1 - i\sqrt{3}}{2}$$

When $(1/27)p^3 + (1/4)q^2$ is negative, A is complex; in this case A should be expressed in trigonometric form: $A = r(\cos\theta + i\sin\theta)$, where θ is a first- or second-quadrant angle, as q is negative or positive. The three roots of the reduced cubic are

$$y_1 = 2(r)^{1/3}\cos(\theta/3)$$

$$y_2 = 2(r)^{1/3}\cos\left(\frac{\theta}{3} + 120°\right)$$

$$y_3 = 2(r)^{1/3}\cos\left(\frac{\theta}{3} + 240°\right)$$

Geometry

Figures 2 to 12 are a collection of common geometric figures. Area (A), volume (V), and other measurable features are indicated.

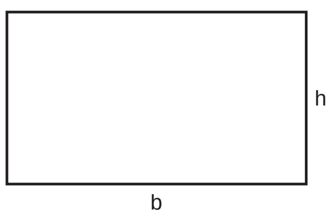


FIGURE 2 Rectangle. $A = bh$.

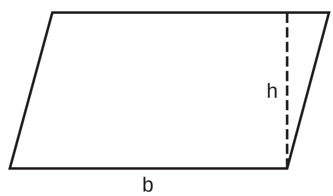


FIGURE 3 Parallelogram. $A = bh$.

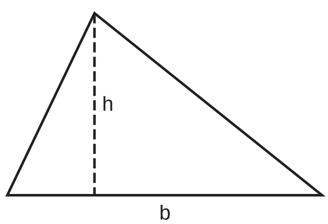


FIGURE 4 Triangle. $A = 1/2\ bh$.

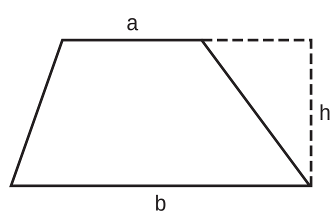


FIGURE 5 Trapezoid. $A = 1/2\ (a + b)h$.

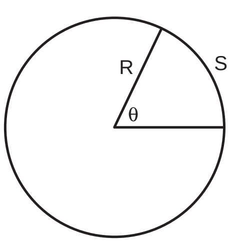


FIGURE 6 Circle. $A = \pi R^2$; circumference $= 2\pi R$; arc length $S = R\theta$ (θ in radians).

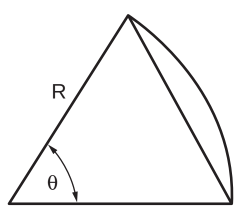


FIGURE 7 Sector of circle. $A_{sector} = 1/2\ R^2\ \theta$; $A_{segment} = 1/2\ R^2\ (\theta - \sin\theta)$.

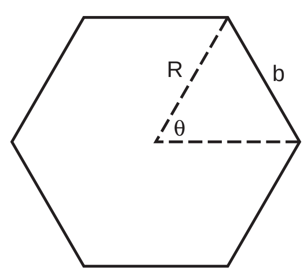


FIGURE 8 Regular polygon of n sides. $A = n/4\ b^2 \operatorname{ctn} \pi/n$; $R = b/2 \csc \pi/n$.

© 2003 by CRC Press LLC

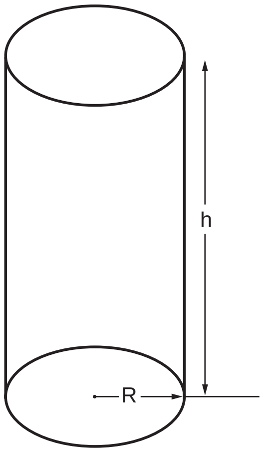


FIGURE 9 Right circular cylinder. $V = \pi R^2 h$; lateral surface area = $2\pi Rh$.

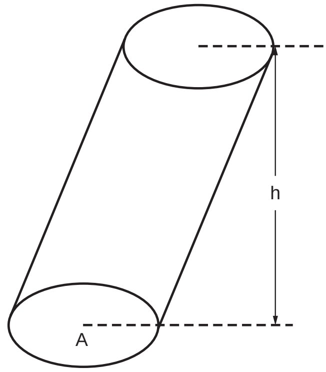


FIGURE 10 Cylinder (or prism) with parallel bases. $V = A/t$.

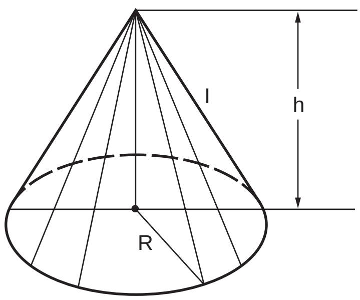


FIGURE 11 Right circular cone. $V = 1/3\ \pi R^2 h$; lateral surface area = $\pi Rl = \pi R\sqrt{R^2 + h^2}$.

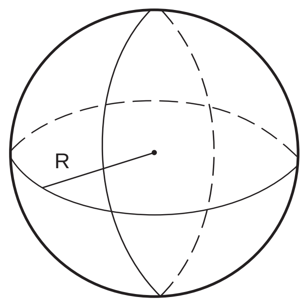


FIGURE 12 Sphere. $V = 4/3\ \pi R^3$; surface area = $4\pi R^2$.

Determinants, Matrices, and Linear Systems of Equations

Determinants

Definition. The square array (matrix) A, with n rows and n columns, has associated with it the determinant

$$\det A = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

a number equal to

$$\sum (\pm) a_{1i} a_{2j} a_{3k} \ldots a_{nl}$$

where $i, j, k, \ldots, l$ is a permutation of the n integers $1, 2, 3, \ldots, n$ in some order. The sign is plus if the permutation is *even* and is minus if the permutation is *odd*. The 2×2 determinant

© 2003 by CRC Press LLC

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

has the value $a_{11}a_{22} - a_{12}a_{21}$ since the permutation (1, 2) is even and (2, 1) is odd. For 3×3 determinants, permutations are as follows:

1,	2,	3	even
1,	3,	2	odd
2,	1,	3	odd
2,	3,	1	even
3,	1,	2	even
3,	2,	1	odd

Thus,

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \left\{ \begin{array}{l} +a_{11} \cdot a_{22} \cdot a_{33} \\ -a_{11} \cdot a_{23} \cdot a_{32} \\ -a_{12} \cdot a_{21} \cdot a_{33} \\ +a_{12} \cdot a_{23} \cdot a_{31} \\ +a_{13} \cdot a_{21} \cdot a_{32} \\ -a_{13} \cdot a_{22} \cdot a_{31} \end{array} \right\}$$

A determinant of order n is seen to be the sum of $n!$ signed products.

Evaluation by Cofactors

Each element a_{ij} has a determinant of order $(n - 1)$ called a *minor* (M_{ij}), obtained by suppressing all elements in row i and column j. For example, the minor of element a_{22} in the 3×3 determinant above is

$$\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}$$

The cofactor of element a_{ij}, denoted A_{ij}, is defined as $\pm M_{ij}$, where the sign is determined from i and j:

$$A_{ij} = (-1)^{i+j} M_{ij}$$

The value of the $n \times n$ determinant equals the sum of products of elements of any row (or column) and their respective cofactors. Thus, for the 3×3 determinant

$$\det A = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} \text{ (first row)}$$

or

$$= a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31} \text{ (first column)}$$

etc.

Properties of Determinants

a. If the corresponding columns and rows of A are interchanged, det A is unchanged.
b. If any two rows (or columns) are interchanged, the sign of det A changes.

© 2003 by CRC Press LLC

c. If any two rows (or columns) are identical, det $A = 0$.
d. If A is triangular (all elements above the main diagonal equal to zero), $A = a_{11} \cdot a_{22} \cdot \ldots \cdot a_{nn}$:

$$\begin{vmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{vmatrix}$$

e. If to each element of a row or column there is added C times the corresponding element in another row (or column), the value of the determinant is unchanged.

Matrices

Definition. A matrix is a rectangular array of numbers and is represented by a symbol A or $[a_{ij}]$:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = [a_{ij}]$$

The numbers a_{ij} are termed *elements* of the matrix; subscripts i and j identify the element as the number in row i and column j. The order of the matrix is $m \times n$ ("m by n"). When $m = n$, the matrix is square and is said to be of order n. For a square matrix of order n, the elements $a_{11}, a_{22}, \ldots, a_{nn}$ constitute the main diagonal.

Operations

Addition. Matrices A and B of the same order may be added by adding corresponding elements, i.e., $A + B = [(a_{ij} + b_{ij})]$.

Scalar multiplication. If $A = [a_{ij}]$ and c is a constant (scalar), then $cA = [ca_{ij}]$, that is, every element of A is multiplied by c. In particular, $(-1)A = -A = [-a_{ij}]$, and $A + (-A) = 0$, a matrix with all elements equal to zero.

Multiplication of matrices. Matrices A and B may be multiplied only when they are conformable, which means that the number of columns of A equals the number of rows of B. Thus, if A is $m \times k$ and B is $k \times n$, then the product $C = AB$ exists as an $m \times n$ matrix with elements c_{ij} equal to the sum of products of elements in row i of A and corresponding elements of column j of B:

$$c_{ij} = \sum_{l=1}^{k} a_{il} b_{lj}$$

For example, if

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & \cdots & \cdots & a_{mk} \end{bmatrix} \cdot \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ b_{k1} & b_{k2} & \cdots & b_{kn} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \cdots & \cdots & \cdots & \\ c_{m1} & c_{m2} & \cdots & c_{mn} \end{bmatrix}$$

then element c_{21} is the sum of products $a_{21}b_{11} + a_{22}b_{21} + \ldots + a_{2k}b_{k1}$.

© 2003 by CRC Press LLC

Properties

$$\begin{aligned} A + B &= B + A \\ A + (B + C) &= (A + B) + C \\ (c_1 + c_2)A &= c_1A + c_2A \\ c(A + B) &= cA + cB \\ c_1(c_2A) &= (c_1c_2)A \\ (AB)(C) &= A(BC) \\ (A + B)(C) &= AC + BC \\ AB &\neq BA \text{ (in general)} \end{aligned}$$

Transpose

If A is an $n \times m$ matrix, the matrix of order $m \times n$ obtained by interchanging the rows and columns of A is called the *transpose* and is denoted A^T. The following are properties of A, B, and their respective transposes:

$$\begin{aligned} (A^T)^T &= A \\ (A + B)^T &= A^T + B^T \\ (cA)^T &= cA^T \\ (AB)^T &= B^TA^T \end{aligned}$$

A *symmetric* matrix is a square matrix A with the property $A = A^T$.

Identity Matrix

A square matrix in which each element of the main diagonal is the same constant a and all other elements are zero is called a *scalar* matrix.

$$\begin{bmatrix} a & 0 & 0 & \cdots & 0 \\ 0 & a & 0 & \cdots & 0 \\ 0 & 0 & a & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \\ 0 & 0 & 0 & \cdots & a \end{bmatrix}$$

When a scalar matrix is multiplied by a conformable second matrix A, the product is aA, which is the same as multiplying A by a scalar a. A scalar matrix with diagonal elements 1 is called the *identity,* or *unit*, matrix and is denoted I. Thus, for any nth-order matrix A, the identity matrix of order n has the property

$$AI = IA = A$$

Adjoint

If A is an n-order square matrix and A_{ij} is the cofactor of element a_{ij}, the transpose of $[A_{ij}]$ is called the *adjoint* of A:

$$adj\,A = [A_{ij}]^T$$

Inverse Matrix

Given a square matrix A of order n, if there exists a matrix B such that $AB = BA = I$, then B is called the *inverse* of A. The inverse is denoted A^{-1}. A necessary and sufficient condition that the square matrix A have an inverse is $\det A \neq 0$. Such a matrix is called *nonsingular;* its inverse is unique and is given by

© 2003 by CRC Press LLC

$$A^{-1} = \frac{adj\,A}{\det A}$$

Thus, to form the inverse of the nonsingular matrix A, form the adjoint of A and divide each element of the adjoint by det A. For example,

$$\begin{bmatrix} 1 & 0 & 2 \\ 3 & -1 & 1 \\ 4 & 5 & 6 \end{bmatrix} \text{ has matrix of cofactors } \begin{bmatrix} -11 & -14 & 19 \\ 10 & -2 & -5 \\ 2 & 5 & -1 \end{bmatrix}$$

$$\text{adjoint} = \begin{bmatrix} -11 & 10 & 2 \\ -14 & -2 & 5 \\ 19 & -5 & -1 \end{bmatrix} \text{ and determinant } = 27$$

Therefore,

$$A^{-1} = \begin{bmatrix} \frac{-11}{27} & \frac{10}{27} & \frac{2}{27} \\ \frac{-14}{27} & \frac{-2}{27} & \frac{5}{27} \\ \frac{19}{27} & \frac{-5}{27} & \frac{-1}{27} \end{bmatrix}$$

Systems of Linear Equations

Given the system

$$\begin{array}{ccccccc} a_{11}x_1 & + & a_{12}x_2 & + \cdots + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + \cdots + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n1}x_1 & + & a_{n2}x_2 & + \cdots + & a_{nn}x_n & = & b_n \end{array}$$

a unique solution exists if det $A \neq 0$, where A is the $n \times n$ matrix of coefficients $[a_{ij}]$.

Solution by Determinants (Cramer's Rule)

$$x_1 = \begin{vmatrix} b_1 & a_{12} & \cdots & a_{1n} \\ b_2 & a_{22} & & \\ \vdots & \vdots & & \vdots \\ b_n & a_{n2} & & a_{nn} \end{vmatrix} \div \det A$$

$$x_2 = \begin{vmatrix} a_{11} & b_1 & a_{13} & \cdots & a_{1n} \\ a_{21} & b_2 & \cdots & & \cdots \\ \vdots & \vdots & & & \\ a_{n1} & b_n & a_{n3} & & a_{nn} \end{vmatrix} \div \det A$$

$$\vdots$$

$$x_k = \frac{\det A_k}{\det A}$$

where A_k is the matrix obtained from A by replacing the kth column of A by the column of bs.

© 2003 by CRC Press LLC

Matrix Solution

The linear system may be written in matrix form $AX = B$, where A is the matrix of coefficients $[a_{ij}]$ and X and B are

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

If a unique solution exists, det $A \neq 0$; hence, A^{-1} exists and

$$X = A^{-1}B$$

Trigonometry

Triangles

In any triangle (in a plane) with sides a, b, and c and corresponding opposite angles A, B, and C,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \qquad \text{(Law of Sines)}$$

$$a^2 = b^2 + c^2 - 2cb\cos A \qquad \text{(Law of Cosines)}$$

$$\frac{a+b}{a-b} = \frac{\tan\frac{1}{2}(A+B)}{\tan\frac{1}{2}(A-B)} \qquad \text{(Law of Tangents)}$$

$$\sin\frac{1}{2}A = \sqrt{\frac{(s-b)(s-c)}{bc}} \qquad \text{where } s = \frac{1}{2}(a+b+c)$$

$$\cos\frac{1}{2}A = \sqrt{\frac{s(s-a)}{bc}}$$

$$\tan\frac{1}{2}A = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$$

$$\begin{aligned} \text{Area} &= \frac{1}{2}bc\sin A \\ &= \sqrt{s(s-a)(s-b)(s-c)} \end{aligned}$$

If the vertices have coordinates (x_1, y_1), (x_2, y_2), and (x_3, y_3), the area is the *absolute value* of the expression

$$\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

© 2003 by CRC Press LLC

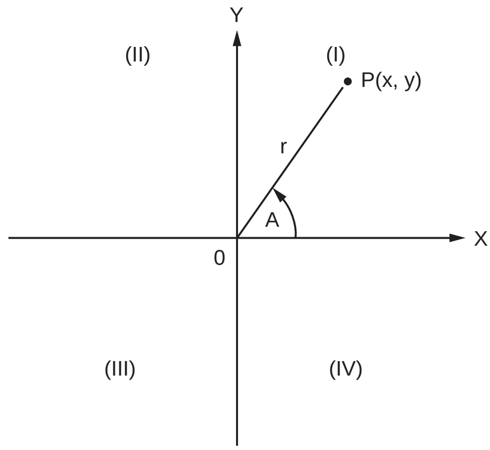


FIGURE 13 The trigonometric point. Angle A is taken to be positive when the rotation is counterclockwise and negative when the rotation is clockwise. The plane is divided into quadrants as shown.

Trigonometric Functions of an Angle

With reference to Figure 13, $P(x, y)$ is a point in either one of the four quadrants and A is an angle whose initial side is coincident with the positive x-axis and whose terminal side contains the point $P(x, y)$. The distance from the origin $P(x, y)$ is denoted by r and is positive. The trigonometric functions of the angle A are defined as

$$\begin{aligned}
\sin A &= \text{sine } A &&= y/r \\
\cos A &= \text{cosine } A &&= x/r \\
\tan A &= \text{tangent } A &&= y/x \\
\text{ctn } A &= \text{cotangent } A &&= x/y \\
\sec A &= \text{secant } A &&= r/x \\
\csc A &= \text{cosecant } A &&= r/y
\end{aligned}$$

z-Transform and the Laplace Transform

When $F(t)$, a continuous function of time, is sampled at regular intervals of period T, the usual Laplace transform techniques are modified. The diagramatic form of a simple sampler, together with its associated input–output waveforms, is shown in Figure 14.

Defining the set of impulse functions $\delta_\tau(t)$ by

$$\delta_\tau(t) \equiv \sum_{n=0}^{\infty} \delta(t - nT)$$

the input–output relationship of the sampler becomes

$$\begin{aligned}
F^*(t) &= F(t) \cdot \delta_\tau(t) \\
&= \sum_{n=0}^{\infty} F(nT) \cdot \delta(t - nT)
\end{aligned}$$

While for a given $F(t)$ and T the $F^*(t)$ is unique, the converse is not true.

© 2003 by CRC Press LLC

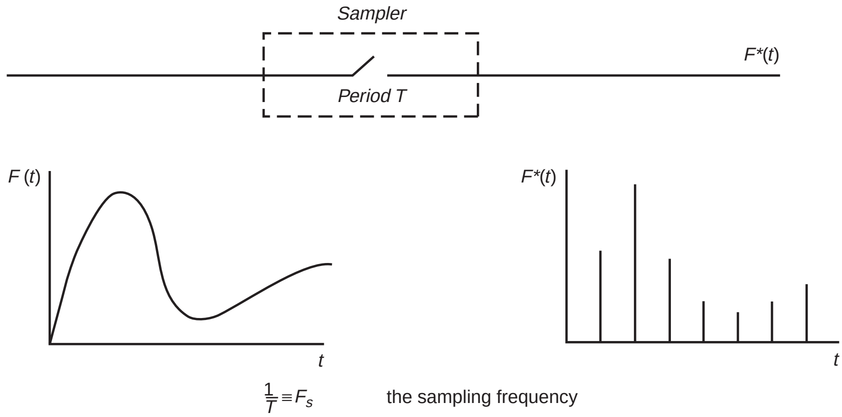


FIGURE 14

For function $U(t)$, the output of the ideal sampler $U^*(t)$ is a set of values $U(kT)$, $k = 0, 1, 2, \ldots$, that is,

$$U^*(t) = \sum_{k=0}^{\infty} U(t)\,\delta(t - kT)$$

The Laplace transform of the output is

$$\mathrm{L} + \{U^*(t)\} = \int_0^{\infty} e^{-st} U^*(t)\,dt = \int_0^{\infty} e^{-st} \sum_{k=0}^{\infty} U(t)\delta(t - kT)\,dt$$

$$= \sum_{k=0}^{\infty} e^{-skT} U(kT)$$

$$\tan A = \frac{1}{\operatorname{ctn} A} = \frac{\sin A}{\cos A}$$

$$\csc A = \frac{1}{\sin A}$$

$$\sec A = \frac{1}{\cos A}$$

$$\operatorname{ctn} A = \frac{1}{\tan A} = \frac{\cos A}{\sin A}$$

$$\sin^2 A + \cos^2 A = 1$$

$$1 + \tan^2 A = \sec^2 A$$

$$1 + \operatorname{ctn}^2 A = \csc^2 A$$

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

© 2003 by CRC Press LLC

$$\sin 2A = 2\sin A \cos A$$

$$\sin 3A = 3\sin A - 4\sin^3 A$$

$$\sin nA = 2\sin(n-1)A\cos A - \sin(n-2)A$$

$$\cos 2A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$$

$$\cos 3A = 4\cos^3 A - 3\cos A$$

$$\cos nA = 2\cos(n-1)A\cos A - \cos(n-2)A$$

$$\sin A + \sin B = 2\sin\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)$$

$$\sin A - \sin B = 2\cos\frac{1}{2}(A+B)\sin\frac{1}{2}(A-B)$$

$$\cos A + \cos B = 2\cos\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)$$

$$\cos A - \cos B = -2\sin\frac{1}{2}(A+B)\sin\frac{1}{2}(A-B)$$

$$\tan A \pm \tan B = \frac{\sin(A \pm B)}{\cos A \cos B}$$

$$\operatorname{ctn} A \pm \operatorname{ctn} B = \pm\frac{\sin(A \pm B)}{\sin A \sin B}$$

$$\sin A \sin B = \frac{1}{2}\cos(A-B) - \frac{1}{2}\cos(A+B)$$

$$\cos A \cos B = \frac{1}{2}\cos(A-B) + \frac{1}{2}\cos(A+B)$$

$$\sin A \cos B = \frac{1}{2}\sin(A+B) + \frac{1}{2}\sin(A-B)$$

$$\sin\frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{2}}$$

$$\cos\frac{A}{2} = \pm\sqrt{\frac{1+\cos A}{2}}$$

$$\tan\frac{A}{2} = \frac{1-\cos A}{\sin A} = \frac{\sin A}{1+\cos A} = \pm\sqrt{\frac{1-\cos A}{1+\cos A}}$$

$$\sin^2 A = \frac{1}{2}(1-\cos 2A)$$

$$\cos^2 A = \frac{1}{2}(1+\cos 2A)$$

© 2003 by CRC Press LLC

$$\sin^3 A = \frac{1}{4}(3\sin A - \sin 3A)$$

$$\cos^3 A = \frac{1}{4}(\cos 3A + 3\cos A)$$

$$\sin ix = \frac{1}{2}i(e^x - e^{-x}) = i\sinh x$$

$$\cos ix = \frac{1}{2}(e^x + e^{-x}) = \cosh x$$

$$\tan ix = \frac{i(e^x - e^{-x})}{e^x + e^{-x}} = i\tanh x$$

$$e^{x+iy} = e^x(\cos y + i\sin y)$$

$$(\cos x \pm i\sin x)^n = \cos nx \pm i\sin nx$$

Inverse Trigonometric Functions

The inverse trigonometric functions are multiple valued, and this should be taken into account in the use of the following formulas.

$$\begin{aligned}
\sin^{-1} x &= \cos^{-1}\sqrt{1-x^2} \\
&= \tan^{-1}\frac{x}{\sqrt{1-x^2}} = \operatorname{ctn}^{-1}\frac{\sqrt{1-x^2}}{x} \\
&= \sec^{-1}\frac{1}{\sqrt{1-x^2}} = \csc^{-1}\frac{1}{x} \\
&= -\sin^{-1}(-x) \\
\cos^{-1} x &= \sin^{-1}\sqrt{1-x^2} \\
&= \tan^{-1}\frac{\sqrt{1-x^2}}{x} = \operatorname{ctn}^{-1}\frac{x}{\sqrt{1-x^2}} \\
&= \sec^{-1}\frac{1}{x} = \csc^{-1}\frac{1}{\sqrt{1-x^2}} \\
&= \pi - \cos^{-1}(-x) \\
\tan^{-1} x &= \operatorname{ctn}^{-1}\frac{1}{x} \\
&= \sin^{-1}\frac{x}{\sqrt{1+x^2}} = \cos^{-1}\frac{1}{\sqrt{1+x^2}} \\
&= \sec^{-1}\sqrt{1+x^2} = \csc^{-1}\frac{\sqrt{1+x^2}}{x} \\
&= -\tan^{-1}(-x)
\end{aligned}$$

© 2003 by CRC Press LLC

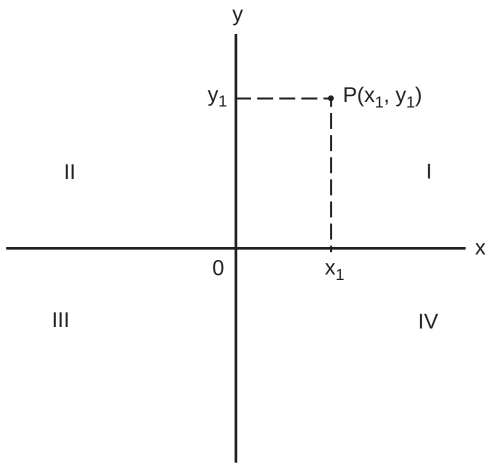


FIGURE 15 Rectangular coordinates.

Analytic Geometry

Rectangular Coordinates

The points in a plane may be placed in one-to-one correspondence with pairs of real numbers. A common method is to use perpendicular lines that are horizontal and vertical and intersect at a point called the *origin.* These two lines constitute the coordinate axes; the horizontal line is the x-axis and the vertical line is the y-axis. The positive direction of the x-axis is to the right, whereas the positive direction of the y-axis is up. If P is a point in the plane, one may draw lines through it that are perpendicular to the x- and y-axes (such as the broken lines of Figure 15). The lines intersect the x-axis at a point with coordinate x_1 and the y-axis at a point with coordinate y_1. We call x_1 the x-coordinate, or *abscissa*, and y_1 is termed the y-coordinate, or *ordinate,* of the point P. Thus, point P is associated with the pair of real numbers (x_1, y_1) and is denoted $P(x_1, y_1)$. The coordinate axes divide the plane into quadrants I, II, III, and IV.

Distance between Two Points; Slope

The distance d between the two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

In the special case when P_1 and P_2 are both on one of the coordinate axes, for instance, the x-axis,

$$d = \sqrt{(x_2 - x_1)^2} = |x_2 - x_1|$$

or on the y-axis,

$$d = \sqrt{(y_2 - y_1)^2} = |y_2 - y_1|$$

The midpoint of the line segment P_1P_2 is

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

© 2003 by CRC Press LLC

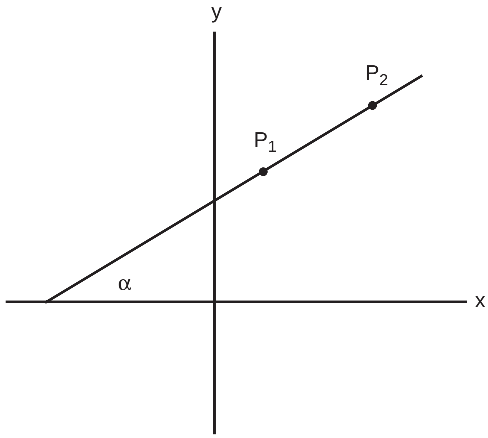


FIGURE 16 The angle of inclination α is the smallest angle measured counterclockwise from the positive x-axis to the line that contains P_1P_2.

The slope of the line segment P_1P_2, provided it is not vertical, is denoted by m and is given by

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

The slope is related to the angle of inclination α (Figure 16) by

$$m = \tan\alpha$$

Two lines (or line segments) with slopes m_1 and m_2 are perpendicular if

$$m_1 = -1/m_2$$

and are parallel if $m_1 = m_2$.

Equations of Straight Lines

A *vertical* line has an equation of the form

$$x = c$$

where $(c, 0)$ is its intersection with the x-axis. A line of slope m through point (x_1, y_1) is given by

$$y - y_1 = m(x - x_1)$$

Thus, a *horizontal line* (slope = 0) through point (x_1, y_1) is given by

$$y = y_1$$

A nonvertical line through the two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is given by either

$$y - y_1 = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1)$$

or

© 2003 by CRC Press LLC

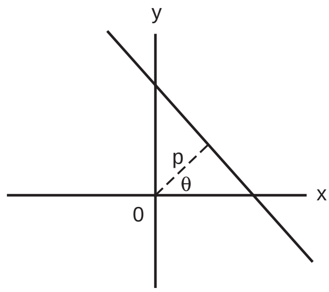


FIGURE 17 Construction for normal form of straight-line equation.

$$y - y_2 = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_2)$$

A line with x-intercept a and y-intercept b is given by

$$\frac{x}{a} + \frac{y}{b} = 1 \qquad (a \neq 0,\ b \neq 0)$$

The *general equation* of a line is

$$Ax + By + C = 0$$

The *normal form* of the straight-line equation is

$$x\cos\theta + y\sin\theta = p$$

where p is the distance along the normal from the origin and θ is the angle that the normal makes with the x-axis (Figure 17).

The general equation of the line $Ax + By + C = 0$ may be written in normal form by dividing by $\pm\sqrt{A^2 + B^2}$, where the plus sign is used when C is negative and the minus sign is used when C is positive:

$$\frac{Ax + By + C}{\pm\sqrt{A^2 + B^2}} = 0$$

so that

$$\cos\theta = \frac{A}{\pm\sqrt{A^2 + B^2}}, \qquad \sin\theta = \frac{B}{\pm\sqrt{A^2 + B^2}}$$

and

$$p = \frac{|C|}{\sqrt{A^2 + B^2}}$$

Distance from a Point to a Line

The perpendicular distance from a point $P(x_1, y_1)$ to the line $Ax + By + C = 0$ is given by

$$d = \frac{Ax_1 + By_1 + C}{\pm\sqrt{A^2 + B^2}}$$

© 2003 by CRC Press LLC

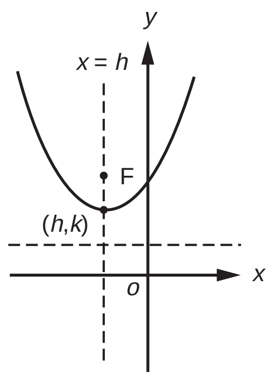


FIGURE 18 Parabola with vertex at (h, k). F identifies the focus.

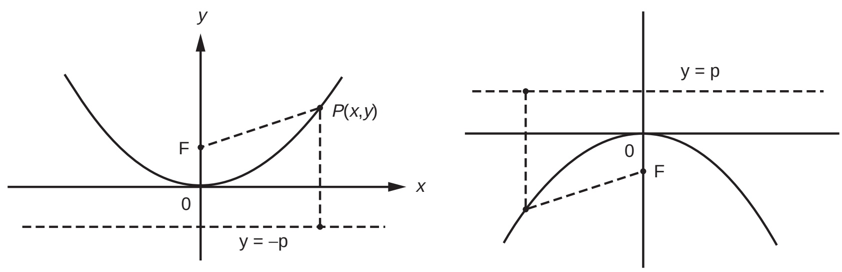


FIGURE 19 Parabolas with y-axis as the axis of symmetry and vertex at the origin. (Left) $y = \frac{x^2}{4p}$; (right) $y = -\frac{x^2}{4p}$.

Circle

The general equation of a circle of radius r and center at $P(x_1, y_1)$ is

$$(x - x_1)^2 + (y - y_1)^2 = r^2$$

Parabola

A parabola is the set of all points (x, y) in the plane that are equidistant from a given line called the *directrix* and a given point called the *focus*. The parabola is symmetric about a line that contains the focus and is perpendicular to the directrix. The line of symmetry intersects the parabola at its *vertex* (Figure 18). The eccentricity $e = 1$.

The distance between the focus and the vertex, or vertex and directrix, is denoted by p (> 0) and leads to one of the following equations of a parabola with vertex at the origin (Figures 19 and 20):

$$y = \frac{x^2}{4p} \qquad \text{(opens upward)}$$

$$y = -\frac{x^2}{4p} \qquad \text{(opens downward)}$$

$$x = \frac{y^2}{4p} \qquad \text{(opens to right)}$$

$$x = -\frac{y^2}{4p} \qquad \text{(opens to left)}$$

© 2003 by CRC Press LLC

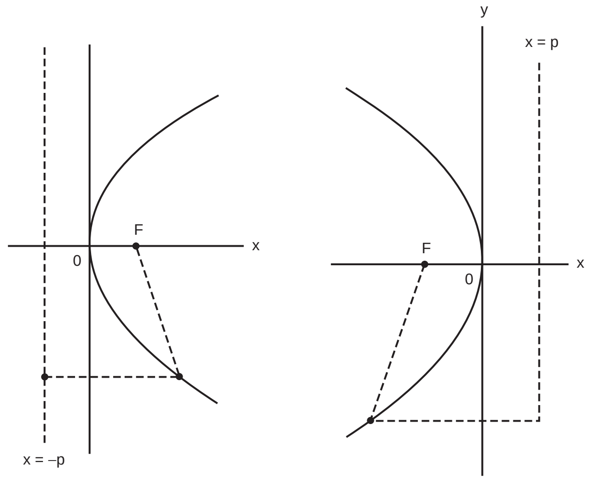


FIGURE 20 Parabolas with x-axis as the axis of symmetry and vertex at the origin. (Left) $x = \frac{y^2}{4p}$; (right) $x = -\frac{y^2}{4p}$.

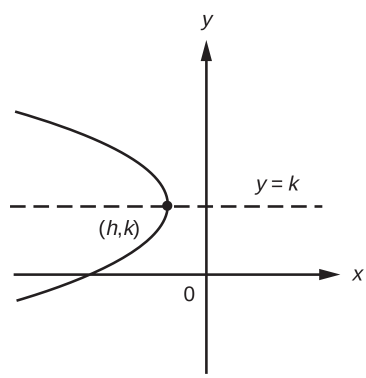


FIGURE 21 Parabola with vertex at (h, k) and axis parallel to the x-axis.

For each of the four orientations shown in Figures 19 and 20, the corresponding parabola with vertex (h, k) is obtained by replacing x by $x - h$ and y by $y - k$. Thus, the parabola in Figure 21 has the equation

$$x - h = -\frac{(y-k)^2}{4p}$$

Ellipse

An ellipse is the set of all points in the plane such that the sum of their distances from two fixed points, called *foci,* is a given constant $2a$. The distance between the foci is denoted $2c$; the length of the major axis is $2a$, whereas the length of the minor axis is $2b$ (Figure 22) and

$$a = \sqrt{b^2 + c^2}$$

The eccentricity of an ellipse, e, is < 1. An ellipse with center at point (h, k) and major axis *parallel to the x-axis* (Figure 23) is given by the equation

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

© 2003 by CRC Press LLC

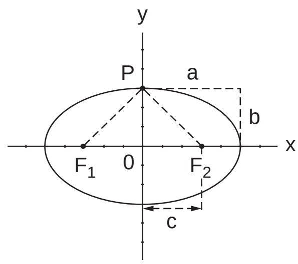


FIGURE 22 Ellipse. Since point P is equidistant from foci F_1 and F_2, the segments F_1P and $F_2P = a$; hence, $a = \sqrt{b^2 + c^2}$.

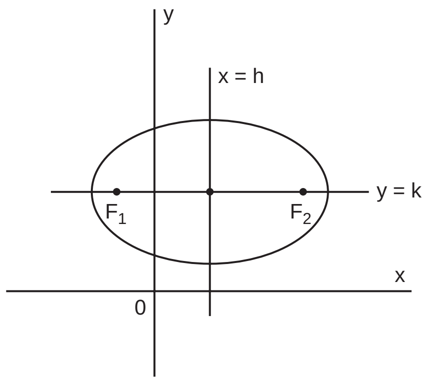


FIGURE 23 Ellipse with major axis parallel to the x-axis. F_1 and F_2 are the foci, each a distance c from center (h, k).

An ellipse with center at (h, k) and major axis parallel to the y-axis is given by the equation (Figure 24)

$$\frac{(y-k)^2}{a^2} + \frac{(x-h)^2}{b^2} = 1$$

Hyperbola ($e > 1$)

A hyperbola is the set of all points in the plane such that the difference of its distances from two fixed points (foci) is a given positive constant denoted $2a$. The distance between the two foci is $2c$ and that between the two vertices is $2a$. The quantity b is defined by the equation

$$b = \sqrt{c^2 - a^2}$$

and is illustrated in Figure 25, which shows the construction of a hyperbola given by the equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

When the focal axis is parallel to the y-axis, the equation of the hyperbola with center (h, k) (Figures 26 and 27) is

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$$

© 2003 by CRC Press LLC

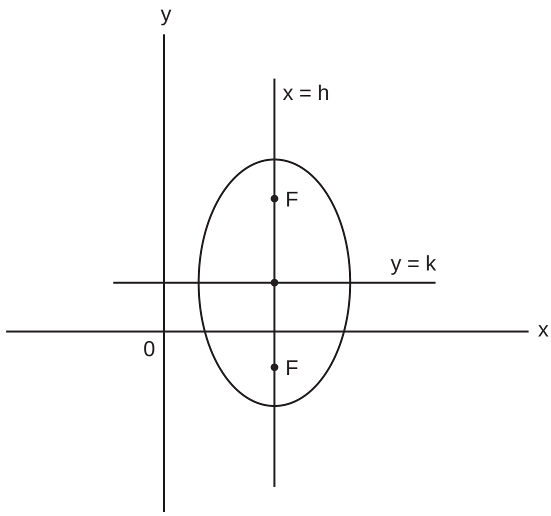


FIGURE 24 Ellipse with major axis parallel to the y-axis. Each focus is a distance c from center (h, k).

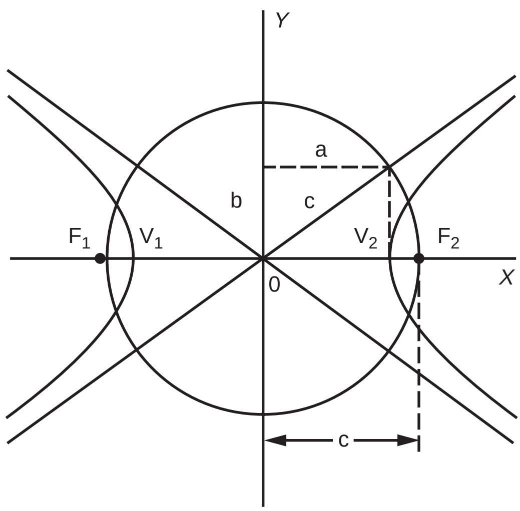


FIGURE 25 Hyperbola. V_1, V_2 = vertices; F_1, F_2 = foci. A circle at center 0 with radius c contains the vertices and illustrates the relation among a, b, and c. Asymptotes have slopes b/a and $-b/a$ for the orientation shown.

If the focal axis is parallel to the x-axis and center (h, k), then

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

Change of Axes

A change in the position of the coordinate axes will generally change the coordinates of the points in the plane. The equation of a particular curve will also generally change.

Translation

When the new axes remain parallel to the original, the transformation is called a *translation* (Figure 28). The new axes, denoted x' and y', have origin $0'$ at (h, k) with reference to the x- and y-axes.

© 2003 by CRC Press LLC

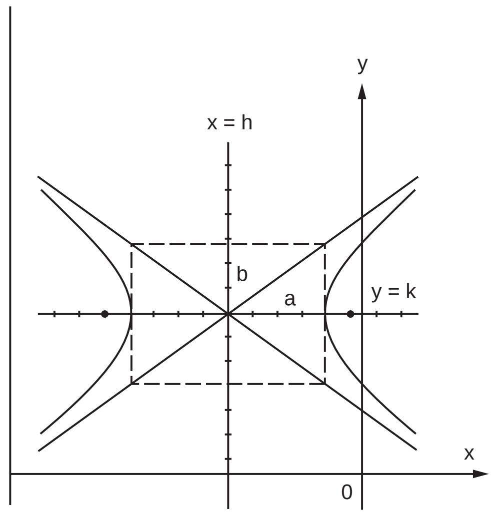


FIGURE 26 Hyperbola with center at (h, k). $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$; slopes of asymptotes $\pm b/a$.

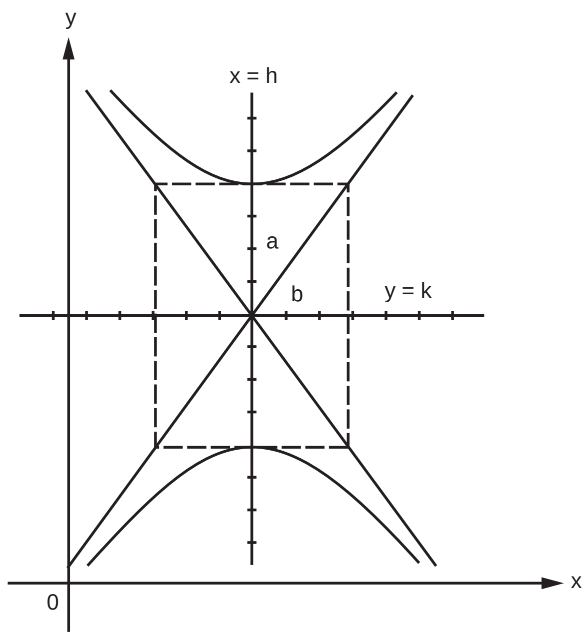


FIGURE 27 Hyperbola with center at (h, k). $\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$; slopes of asymptotes $\pm a/b$.

Series

Bernoulli and Euler Numbers

A set of numbers, $B_1, B_3, \ldots, B_{2n-1}$ (Bernoulli numbers) and $B_2, B_4, \ldots, B_{2n}$ (Euler numbers), appears in the series expansions of many functions. A partial listing follows; these are computed from the following equations:

© 2003 by CRC Press LLC

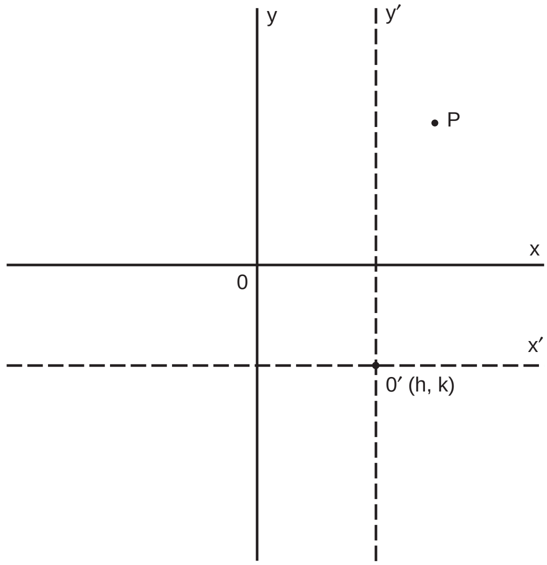


FIGURE 28 Translation of axes.

$$B_{2n} - \frac{2n(2n-1)}{2!}B_{2n-2} + \frac{2n(2n-1)(2n-2)(2n-3)}{4!}B_{2n-4} - \cdots + (-1)^n = 0$$

and

$$\frac{2^{2n}(2^{2n}-1)}{2n}B_{2n-1} = (2n-1)B_{2n-2} - \frac{(2n-1)(2n-2)(2n-3)}{3!}B_{2n-4} + \cdots + (-1)^{n-1}$$

$$\begin{aligned}
B_1 &= 1/6 & B_2 &= 1 \\
B_3 &= 1/30 & B_4 &= 5 \\
B_5 &= 1/42 & B_6 &= 61 \\
B_7 &= 1/30 & B_8 &= 1385 \\
B_9 &= 5/66 & B_{10} &= 50{,}521
\end{aligned}$$

$$\begin{aligned}
B_{11} &= 691/2730 & B_{12} &= 2{,}702{,}765 \\
B_{13} &= 7/6 & B_{14} &= 199{,}360{,}981 \\
&\vdots & &\vdots
\end{aligned}$$

Series of Functions

In the following, the interval of convergence is indicated; otherwise, it is all x. Logarithms are of base e. Bernoulli and Euler numbers (B_{2n-1} and B_{2n}) appear in certain expressions.

$$(a+x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{2!}a^{n-2}x^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}x^3 + \cdots \qquad [x^2 < a^2]$$
$$+ \frac{n!}{(n-j)!j!}a^{n-j}x^j + \cdots$$

© 2003 by CRC Press LLC

$$(a-bx)^{-1} = \frac{1}{a}\left[1+\frac{bx}{a}+\frac{b^2x^2}{a^2}+\frac{b^3x^3}{a^3}+\cdots\right] \qquad [b^2x^2<a^2]$$

$$(1\pm x)^n = 1 \pm nx + \frac{n(n-1)}{2!}x^2 \pm \frac{n(n-1)(n-2)x^3}{3!} + \cdots \qquad [x^2<1]$$

$$(1\pm x)^{-n} = 1 \mp nx + \frac{n(n+1)}{2!}x^2 \mp \frac{n(n+1)(n+2)}{3!}x^3 + \cdots \qquad [x^2<1]$$

$$(1\pm x)^{\frac{1}{2}} = 1 \pm \frac{1}{2}x - \frac{1}{2\cdot 4}x^2 \pm \frac{1\cdot 3}{2\cdot 4\cdot 6}x^3 - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 8}x^4 \pm \cdots \qquad [x^2<1]$$

$$(1\pm x)^{-\frac{1}{2}} = 1 \mp \frac{1}{2}x + \frac{1\cdot 3}{2\cdot 4}x^2 \mp \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^3 + \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}x^4 \mp \cdots \qquad [x^2<1]$$

$$(1\pm x^2)^{\frac{1}{2}} = 1 \pm \frac{1}{2}x^2 - \frac{x^4}{2\cdot 4} \pm \frac{1\cdot 3}{2\cdot 4\cdot 6}x^6 - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 8}x^8 \pm \cdots \qquad [x^2<1]$$

$$(1\pm x)^{-1} = 1 \mp x + x^2 \mp x^3 + x^4 \mp x^5 + \cdots \qquad [x^2<1]$$

$$(1\pm x)^{-2} = 1 \mp 2x + 3x^2 \mp 4x^3 + 5x^4 \mp \cdots \qquad [x^2<1]$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} - \cdots$$

$$a^x = 1 + x\log a + \frac{(x\log a)^2}{2!} + \frac{(x\log a)^3}{3!} + \cdots$$

$$\log x = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \cdots \qquad [0 < x < 2]$$

$$\log x = \frac{x-1}{x} + \frac{1}{2}\left(\frac{x-1}{x}\right)^2 + \frac{1}{3}\left(\frac{x-1}{x}\right)^3 + \cdots \qquad \left[x>\frac{1}{2}\right]$$

$$\log x = 2\left[\left(\frac{x-1}{x+1}\right) + \frac{1}{3}\left(\frac{x-1}{x+1}\right)^3 + \frac{1}{5}\left(\frac{x-1}{x+1}\right)^5 + \cdots\right] \qquad [x > 0]$$

$$\log(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots \qquad [x^2<1]$$

$$\log\left(\frac{1+x}{1-x}\right) = 2\left[x + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \frac{1}{7}x^7 + \cdots\right] \qquad [x^2<1]$$

$$\log\left(\frac{x+1}{x-1}\right) = 2\left[\frac{1}{x} + \frac{1}{3}\left(\frac{1}{x}\right)^3 + \frac{1}{5}\left(\frac{1}{x}\right)^5 + \cdots\right] \qquad [x^2>1]$$

© 2003 by CRC Press LLC

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \cdots + \frac{2^{2n}(2^{2n}-1)B_{2n-1}x^{2n-1}}{(2n)!} \qquad \left[x^2 < \frac{\pi^2}{4}\right]$$

$$\operatorname{ctn} x = \frac{1}{x} - \frac{x}{3} - \frac{x^3}{45} - \frac{2x^5}{945} - \cdots - \frac{B_{2n-1}(2x)^{2n}}{(2n)!x} - \cdots \qquad [x^2 < \pi^2]$$

$$\sec x = 1 + \frac{x^2}{2!} + \frac{5x^4}{4!} + \frac{61x^6}{6!} + \cdots + \frac{B_{2n}x^{2n}}{(2n)!} + \cdots \qquad \left[x^2 < \frac{\pi^2}{4}\right]$$

$$\csc x = \frac{1}{x} + \frac{x}{3!} + \frac{7x^3}{3\cdot 5!} + \frac{31x^5}{3\cdot 7!} + \cdots + \frac{2(2^{2n+1}-1)}{(2n+2)!}B_{2n+1}x^{2n+1} + \cdots \qquad [x^2 < \pi^2]$$

$$\sin^{-1} x = x + \frac{x^3}{6} + \frac{(1\cdot 3)x^5}{(2\cdot 4)5} + \frac{(1\cdot 3\cdot 5)x^7}{(2\cdot 4\cdot 6)7} + \cdots \qquad [x^2 < 1]$$

$$\tan^{-1} x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \cdots \qquad [x^2 < 1]$$

$$\sec^{-1} x = \frac{\pi}{2} - \frac{1}{x} - \frac{1}{6x^3} - \frac{1\cdot 3}{(2\cdot 4)5x^5} - \frac{1\cdot 3\cdot 5}{(2\cdot 4\cdot 6)7x^7} - \cdots \qquad [x^2 > 1]$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \frac{x^8}{8!} + \cdots$$

$$\tanh x = (2^2-1)2^2B_1\frac{x}{2!} - (2^4-1)2^4B_3\frac{x^3}{4!} + (2^6-1)2^6B_5\frac{x^5}{6!} - \cdots \qquad \left[x^2 < \frac{\pi^2}{4}\right]$$

$$\operatorname{ctnh} x = \frac{1}{x}\left(1 + \frac{2^2B_1x^2}{2!} - \frac{2^4B_3x^4}{4!} + \frac{2^6B_5x^6}{6!} - \cdots\right) \qquad [x^2 < \pi^2]$$

$$\operatorname{sech} x = 1 - \frac{B_2x^2}{2!} + \frac{B_4x^4}{4!} - \frac{B_6x^6}{6!} + \cdots \qquad \left[x^2 < \frac{\pi^2}{4}\right]$$

$$\operatorname{csch} x = \frac{1}{x} - (2-1)2B_1\frac{x}{2!} + (2^3-1)2B_3\frac{x^3}{4!} - \cdots \qquad [x^2 < \pi^2]$$

$$\sinh^{-1} x = x - \frac{1}{2}\frac{x^3}{3} + \frac{1\cdot 3}{2\cdot 4}\frac{x^5}{5} - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{x^7}{7} + \cdots \qquad [x^2 < 1]$$

$$\tanh^{-1} x = x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \cdots \qquad [x^2 < 1]$$

© 2003 by CRC Press LLC

$$\operatorname{ctnh}^{-1} x = \frac{1}{x} + \frac{1}{3x^3} + \frac{1}{5x^5} + \cdots \qquad [x^2 > 1]$$

$$\operatorname{csch}^{-1} x = \frac{1}{x} - \frac{1}{2 \cdot 3x^3} + \frac{1 \cdot 3}{2 \cdot 4 \cdot 5x^5} - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 7x^7} + \cdots \qquad [x^2 > 1]$$

$$\int_0^x e^{-t^2} dt = x - \frac{1}{3}x^3 + \frac{x^5}{5 \cdot 2!} - \frac{x^7}{7 \cdot 3!} + \cdots$$

Error Function

The following function, known as the error function, erf x, arises frequently in applications:

$$\operatorname{erf} x = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$

The integral cannot be represented in terms of a finite number of elementary functions; therefore, values of erf x have been compiled in tables. The following is the series for erf x.

$$\operatorname{erf} x = \frac{2}{\sqrt{\pi}} \left[x - \frac{x^3}{3} + \frac{x^5}{5 \cdot 2!} - \frac{x^7}{7 \cdot 3!} + \cdots \right]$$

There is a close relation between this function and the area under the standard normal curve (Table 1 in the Tables of Probability and Statistics). For evaluation, it is convenient to use z instead of x; then erf z may be evaluated from the area $F(z)$ given in Table 1 by use of the relation

$$\operatorname{erf} z = 2F(\sqrt{2}z)$$

Example

$$\operatorname{erf}(0.5) = 2F[(1.414)(0.5)] = 2F(0.707)$$

By interpolation from Table 1, $F(0.707) = 0.260$; thus, $\operatorname{erf}(0.5) = 0.520$.

Series Expansion

The expression in parentheses following certain of the series indicates the region of convergence. If not otherwise indicated, it is to be understood that the series converges for all finite values of x.

Binomial

$$(x + y)^n = x^n + nx^{n-1}y + \frac{n(n-1)}{2!}x^{n-2}y^2 + \frac{n(n-1)(n-2)}{3!}x^{n-3}y^3 + \cdots \qquad (y^2 < x^2)$$

$$(1 \pm x)^n = 1 \pm nx + \frac{n(n-1)x^2}{2!} \pm \frac{n(n-1)(n-2)x^3}{3!} + \cdots \text{ etc.} \qquad (x^2 < 1)$$

$$(1 \pm x)^{-n} = 1 \mp nx + \frac{n(n+1)x^2}{2!} \mp \frac{n(n+1)(n+2)x^3}{3!} + \cdots \text{ etc.} \qquad (x^2 < 1)$$

$$(1 \pm x)^{-1} = 1 \mp x + x^2 \mp x^3 + x^4 \mp x^5 + \cdots \qquad (x^2 < 1)$$

$$(1 \pm x)^{-2} = 1 \mp 2x + 3x^2 \mp 4x^3 + 5x^4 \mp 6x^5 + \cdots \qquad (x^2 < 1)$$

© 2003 by CRC Press LLC

Reversion of Series

Let a series be represented by

$$y = a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + \cdots \qquad (a_1 \neq 0)$$

to find the coefficients of the series

$$x = A_1 y + A_2 y^2 + A_3 y^3 + A_4 y^4 + \cdots$$

$$A_1 = \frac{1}{a_1} \qquad A_2 = -\frac{a_2}{a_1^3} \qquad A_3 = \frac{1}{a_1^5}(2a_2^2 - a_1 a_3)$$

$$A_4 = \frac{1}{a_1^7}(5a_1 a_2 a_3 - a_1^2 a_4 - 5a_2^3)$$

$$A_5 = \frac{1}{a_1^9}(6a_1^2 a_2 a_4 + 3a_1^2 a_3^2 + 14a_2^4 - a_1^3 a_5 - 21 a_1 a_2^2 a_3)$$

$$A_6 = \frac{1}{a_1^{11}}(7a_1^3 a_2 a_5 + 7a_1^3 a_3 a_4 + 84 a_1 a_2^3 a_3 - a_1^4 a_6 - 28 a_1^2 a_2^2 a_4 - 28 a_1^2 a_2 a_3^2 - 42 a_2^5)$$

$$A_7 = \frac{1}{a_1^{13}}(8a_1^4 a_2 a_6 + 8a_1^4 a_3 a_5 + 4a_1^4 a_4^2 + 120 a_1^2 a_2^3 a_4 + 180 a_1^2 a_2^2 a_3^2 + 132 a_2^6 - a_1^5 a_7$$
$$- 36 a_1^3 a_2^2 a_5 - 72 a_1^3 a_2 a_3 a_4 - 12 a_1^3 a_3^3 - 330 a_1 a_2^4 a_3)$$

Taylor

1. $$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a)$$
$$+ \cdots + \frac{(x-a)^n}{n!}f^{(n)}(a) + \cdots \text{(Taylor's series)}$$

(Increment form)

2. $$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \cdots$$
$$= f(h) + xf'(h) + \frac{x^2}{2!}f''(h) + \frac{x^3}{3!}f'''(h) + \cdots$$

3. If $f(x)$ is a function possessing derivatives of all orders throughout the interval $a \leqq x \leqq b$, then there is a value X, with $a < X < b$, such that

$$f(b) = f(a) + (b-a)f'(a) + \frac{(b-a)^2}{2!}f''(a) + \cdots$$
$$+ \frac{(b-a)^{n-1}}{(n-1)!}f^{(n-1)}(a) + \frac{(b-a)^n}{n!}f^{(n)}(X)$$

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \cdots + \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a)$$
$$+ \frac{h^n}{n!}f^{(n)}(a + \theta h), \qquad b = a + h, 0 < \theta < 1$$

© 2003 by CRC Press LLC

or

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots + (x-a)^{n-1}\frac{f^{(n-1)}(a)}{(n-1)!} + R_n$$

where

$$R_n = \frac{f^{(n)}[a + \theta \cdot (x-a)]}{n!}(x-a)^n, \; 0 < \theta < 1$$

The above forms are known as Taylor's series with the remainder term.

4. Taylor's series for a function of two variables:

$$\text{If} \quad \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)f(x,y) = h\frac{\partial f(x,y)}{\partial x} + k\frac{\partial f(x,y)}{\partial y}$$

$$\text{and} \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(x,y) = h^2\frac{\partial^2 f(x,y)}{\partial x^2} + 2hk\frac{\partial^2 f(x,y)}{\partial x \partial y} + k^2\frac{\partial^2 f(x,y)}{\partial y^2}pp$$

etc., and if $\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f(x,y)\Big|_{\substack{x=a\\y=b}}$ with the bar and subscripts means that after differentiation we are to replace x by a and y by b,

$$\text{then } f(a+h, b+k) = f(a,b) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)f(x,y)\Big|_{\substack{x=a\\y=b}} + \cdots$$

$$+ \frac{1}{n!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f(x,y)\Big|_{\substack{x=a\\y=b}} + \cdots$$

MacLaurin

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots + x^{n-1}\frac{f^{(n-1)}(0)}{(n-1)!} + R_n$$

where

$$R_n = \frac{x^n f^{(n)}(\theta x)}{n!}, \quad 0 < \theta < 1$$

Exponential

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \qquad \text{(all real values of } x\text{)}$$

$$a^x = 1 + x\log_e a + \frac{(x\log_e a)^2}{2!} + \frac{(x\log_e a)^3}{3!} + \cdots$$

$$e^x = e^a\left[1 + (x-a) + \frac{(x-a)^2}{2!} + \frac{(x-a)^3}{3!} + \cdots\right]$$

© 2003 by CRC Press LLC

Logarithmic

$$\log_e x = \frac{x-1}{x} + \frac{1}{2}\left(\frac{x-1}{x}\right)^2 + \frac{1}{3}\left(\frac{x-1}{x}\right)^3 + \cdots \qquad (x > \tfrac{1}{2})$$

$$\log_e x = (x-1) - \tfrac{1}{2}(x-1)^2 + \tfrac{1}{3}(x-1)^3 - \cdots \qquad (2 \geq x > 0)$$

$$\log_e x = 2\left[\frac{x-1}{x+1} + \frac{1}{3}\left(\frac{x-1}{x+1}\right)^3 + \frac{1}{5}\left(\frac{x-1}{x+1}\right)^5 + \cdots\right] \qquad (x > 0)$$

$$\log_e(1+x) = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 + \cdots \qquad (-1 < x \leq 1)$$

$$\log_e(n+1) - \log_e(n-1) = 2\left[\frac{1}{n} + \frac{1}{3n^3} + \frac{1}{5n^5} + \cdots\right]$$

$$\log_e(a+x) = \log_e a + 2\left[\frac{x}{2a+x} + \frac{1}{3}\left(\frac{x}{2a+x}\right)^3 + \frac{1}{5}\left(\frac{x}{2a+x}\right)^5 + \cdots\right]$$

$$(a > 0, -a < x < +\infty)$$

$$\log_e \frac{1+x}{1-x} = 2\left[x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots + \frac{x^{2n-1}}{2n-1} + \cdots\right] \qquad (-1 < x < 1)$$

$$\log_e x = \log_e a + \frac{(x-a)}{a} - \frac{(x-a)^2}{2a^2} + \frac{(x-a)^3}{3a^3} - \quad + \cdots \qquad (0 < x \leqq 2a)$$

Trigonometric

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad \text{(all real values of } x\text{)}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \qquad \text{(all real values of } x\text{)}$$

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \frac{62x^9}{2835} + \cdots$$

$$+ \frac{(-1)^{n-1}2^{2n}(2^{2n}-1)B_{2n}}{(2n)!}x^{2n-1} + \cdots \qquad \left[x^2 < \frac{\pi^2}{4}, \text{ and } B_n \text{ represents the } n\text{th Bernoulli number}\right]$$

$$\cot x = \frac{1}{x} - \frac{x}{3} - \frac{x^2}{45} - \frac{2x^5}{945} - \frac{x^7}{4725} - \cdots$$

$$- \frac{(-1)^{n+1}2^{2n}}{(2n)!}B_{2n}x^{2n-1} - \cdots \qquad \left[x^2 < \pi^2, \text{ and } B_n \text{ represents the } n\text{th Bernoulli number}\right]$$

© 2003 by CRC Press LLC

Differential Calculus

Notation

For the following equations, the symbols $f(x)$, $g(x)$, etc. represent functions of x. The value of a function $f(x)$ at $x = a$ is denoted $f(a)$. For the function $y = f(x)$, the derivative of y with respect to x is denoted by one of the following:

$$\frac{dy}{dx}, \quad f'(x), \quad D_x y, \quad y'$$

Higher derivatives are as follows:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}f'(x) = f''(x)$$

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d}{dx}f''(x) = f'''(x), \text{ etc.}$$

and values of these at $x = a$ are denoted $f''(a)$, $f'''(a)$, etc. (see Table of Derivatives).

Slope of a Curve

The tangent line at a point $P(x, y)$ of the curve $y = f(x)$ has a slope $f'(x)$, provided that $f'(x)$ exists at P. The slope at P is defined to be that of the tangent line at P. The tangent line at $P(x_1, y_1)$ is given by

$$y - y_1 = f'(x_1)(x - x_1)$$

The *normal line* to the curve at $P(x_1, y_1)$ has slope $-1/f'(x_1)$ and thus obeys the equation

$$y - y_1 = [-1/f'(x_1)](x - x_1)$$

(The slope of a vertical line is not defined.)

Angle of Intersection of Two Curves

Two curves, $y = f_1(x)$ and $y = f_2(x)$, that intersect at a point $P(X, Y)$ where derivatives $f_1'(X)$, $f_2'(X)$ exist have an angle (α) of intersection given by

$$\tan\alpha = \frac{f_2'(X) - f_1'(X)}{1 + f_2'(X) \cdot f_1'(X)}$$

If $\tan\alpha > 0$, then α is the acute angle; if $\tan\alpha < 0$, then α is the obtuse angle.

Radius of Curvature

The radius of curvature R of the curve $y = f(x)$ at point $P(x, y)$ is

$$R = \frac{\{1 + [f'(x)]^2\}^{3/2}}{f''(x)}$$

In polar coordinates (θ, r), the corresponding formula is

© 2003 by CRC Press LLC

$$R = \frac{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]^{3/2}}{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}}$$

The *curvature* K is $1/R$.

Relative Maxima and Minima

The function f has a relative maximum at $x = a$ if $f(a) \geq f(a + c)$ for all values of c (positive or negative) that are sufficiently near zero. The function f has a relative minimum at $x = b$ if $f(b) \leq f(b + c)$ for all values of c that are sufficiently close to zero. If the function f is defined on the closed interval $x_1 \leq x \leq x_2$ and has a relative maximum or minimum at $x = a$, where $x_1 < a < x_2$, and if the derivative $f'(x)$ exists at $x = a$, then $f'(a) = 0$. It is noteworthy that a relative maximum or minimum may occur at a point where the derivative does not exist. Further, the derivative may vanish at a point that is neither a maximum nor a minimum for the function. Values of x for which $f'(x) = 0$ are called "critical values." To determine whether a critical value of x, say x_c, is a relative maximum or minimum for the function at x_c, one may use the second derivative test:

1. If $f''(x_c)$ is positive, $f(x_c)$ is a minimum.
2. If $f''(x_c)$ is negative, $f(x_c)$ is a maximum.
3. If $f''(x_c)$ is zero, no conclusion may be made.

The sign of the derivative as x advances through x_c may also be used as a test. If $f'(x)$ changes from positive to zero to negative, then a maximum occurs at x_c, whereas a change in $f'(x)$ from negative to zero to positive indicates a minimum. If $f'(x)$ does not change sign as x advances through x_c, then the point is neither a maximum nor a minimum.

Points of Inflection of a Curve

The sign of the second derivative of f indicates whether the graph of $y = f(x)$ is concave upward or concave downward:

$$f''(x) > 0: \quad \text{concave upward}$$

$$f''(x) < 0: \quad \text{concave downward}$$

A point of the curve at which the direction of concavity changes is called a point of inflection (Figure 29). Such a point may occur where $f''(x) = 0$ or where $f''(x)$ becomes infinite. More precisely, if the function $y = f(x)$ and its first derivative $y' = f'(x)$ are continuous in the interval $a \leq x \leq b$, and if $y'' = f''(x)$ exists in $a < x < b$, then the graph of $y = f(x)$ for $a < x < b$ is concave upward if $f''(x)$ is positive and concave downward if $f''(x)$ is negative.

Taylor's Formula

If f is a function that is continuous on an interval that contains a and x, and if its first $(n + 1)$ derivatives are continuous on this interval, then

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R$$

where R is called the *remainder*. There are various common forms of the remainder:

© 2003 by CRC Press LLC

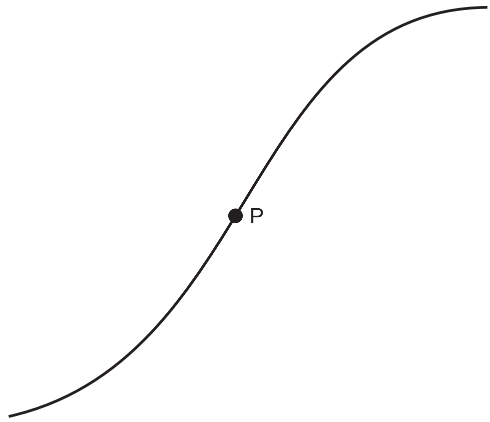


FIGURE 29 Point of inflection.

Lagrange's Form

$$R = f^{(n+1)}(\beta) \cdot \frac{(x-a)^{n+1}}{(n+1)!}; \ \beta \text{ between } a \text{ and } x$$

Cauchy's Form

$$R = f^{(n+1)}(\beta) \cdot \frac{(x-\beta)^n (x-a)}{n!}; \ \beta \text{ between } a \text{ and } x$$

Integral Form

$$R = \int_a^x \frac{(x-t)^n}{n!} f^{(n+1)}(t)\, dt$$

Indeterminant Forms

If $f(x)$ and $g(x)$ are continuous in an interval that includes $x = a$, and if $f(a) = 0$ and $g(a) = 0$, the limit $\lim_{x \to a} (f(x)/g(x))$ takes the form "0/0," called an *indeterminant form*. *L'Hôpital's rule* is

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

Similarly, it may be shown that if $f(x) \to \infty$ and $g(x) \to \infty$ as $x \to a$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

(The above holds for $x \to \infty$.)

Examples

$$\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{\cos x}{1} = 1$$

$$\lim_{x \to \infty} \frac{x^2}{e^x} = \lim_{x \to \infty} \frac{2x}{e^x} = \lim_{x \to \infty} \frac{2}{e^x} = 0$$

© 2003 by CRC Press LLC

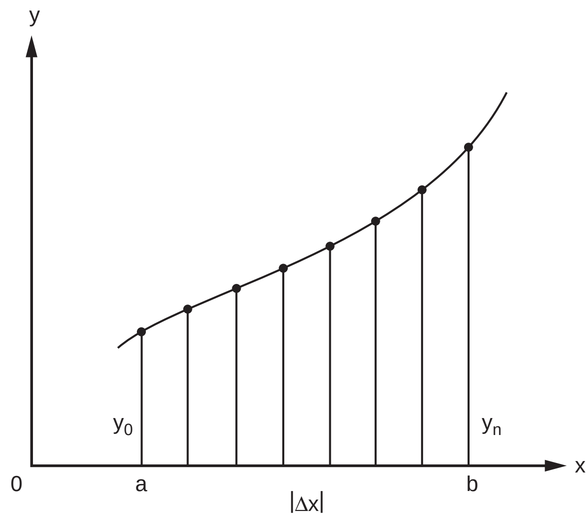


FIGURE 30 Trapezoidal rule for area.

Numerical Methods

a. *Newton's method* for approximating roots of the equation $f(x) = 0$: A first estimate x_1 of the root is made; then, provided that $f'(x_1) \neq 0$, a better approximation is x_2:

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

The process may be repeated to yield a third approximation x_3 to the root:

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

provided $f'(x_2)$ exists. The process may be repeated. (In certain rare cases, the process will not converge.)

b. *Trapezoidal rule for areas* (Figure 30): For the function $y = f(x)$ defined on the interval (a, b) and positive there, take n equal subintervals of width $\Delta x = (b - a)/n$. The area bounded by the curve between $x = a$ and $x = b$ (or definite integral of $f(x)$) is approximately the sum of trapezoidal areas, or

$$A \sim \left(\frac{1}{2}y_0 + y_1 + y_2 + \cdots + y_{n-1} + \frac{1}{2}y_n\right)(\Delta x)$$

Estimation of the error (E) is possible if the second derivative can be obtained:

$$E = \frac{b-a}{12} f''(c)(\Delta x)^2$$

where c is some number between a and b.

Functions of Two Variables

For the function of two variables, denoted $z = f(x, y)$, if y is held constant, say at $y = y_1$, then the resulting function is a function of x only. Similarly, x may be held constant at x_1, to give the resulting function of y.

© 2003 by CRC Press LLC

The Gas Laws

A familiar example is afforded by the ideal gas law that relates the pressure p, the volume V, and the absolute temperature T of an ideal gas:

$$pV = nRT$$

where n is the number of moles and R is the gas constant per mole, 8.31 ($J \cdot K^{-1} \cdot mole^{-1}$). By rearrangement, any one of the three variables may be expressed as a function of the other two. Further, either one of these two may be held constant. If T is held constant, then we get the form known as Boyle's law:

$$p = kV^{-1} \qquad \text{(Boyle's law)}$$

where we have denoted nRT by the constant k and, of course, $V > 0$. If the pressure remains constant, we have Charles' law:

$$V = bT \qquad \text{(Charles' law)}$$

where the constant b denotes nR/p. Similarly, volume may be kept constant:

$$p = aT$$

where now the constant, denoted a, is nR/V.

Partial Derivatives

The physical example afforded by the ideal gas law permits clear interpretations of processes in which one of the variables is held constant. More generally, we may consider a function $z = f(x, y)$ defined over some region of the x–y-plane in which we hold one of the two coordinates, say y, constant. If the resulting function of x is differentiable at a point (x, y), we denote this derivative by one of the notations

$$f_x, \qquad \delta f/dx, \qquad \delta z/dx$$

called the *partial derivative with respect to x*. Similarly, if x is held constant and the resulting function of y is differentiable, we get the *partial derivative with respect to y*, denoted by one of the following:

$$f_y, \qquad \delta f/dy, \qquad \delta z/dy$$

Example

$$\begin{aligned} \text{Given } z &= x^4y^3 - y \sin x + 4y, \text{ then} \\ \delta z/dx &= 4(xy)^3 - y \cos x \\ \delta z/dy &= 3x^4y^2 - \sin x + 4 \end{aligned}$$

Integral Calculus

Indefinite Integral

If $F(x)$ is differentiable for all values of x in the interval (a, b) and satisfies the equation $dy/dx = f(x)$, then $F(x)$ is an integral of $f(x)$ with respect to x. The notation is $F(x) = \int f(x)\, dx$ or, in differential form, $dF(x) = f(x)\,dx$.

For any function $F(x)$ that is an integral of $f(x)$, it follows that $F(x) + C$ is also an integral. We thus write

$$\int f(x)dx = F(x) + C$$

© 2003 by CRC Press LLC

Definite Integral

Let $f(x)$ be defined on the interval $[a, b]$ which is partitioned by points $x_1, x_2, \ldots, x_j, \ldots, x_{n-1}$ between $a = x_0$ and $b = x_n$. The jth interval has length $\Delta x_j = x_j - x_{j-1}$, which may vary with j. The sum $\Sigma_{j=1}^{n} f(\upsilon_j)\Delta x_j$, where υ_j is arbitrarily chosen in the jth subinterval, depends on the numbers $x_0, \ldots, x_n$ and the choice of the υ as well as f; however, if such sums approach a common value as all Δx approach zero, then this value is the definite integral of f over the interval (a, b) and is denoted $\int_a^b f(x)\,dx$. The *fundamental theorem of integral calculus* states that

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where F is any continuous indefinite integral of f in the interval (a, b).

Properties

$$\int_a^b [f_1(x) + f_2(x) + \cdots + f_j(x)]\,dx = \int_a^b f_1(x)\,dx + \int_a^b f_2(x)\,dx + \cdots + \int_a^b f_j(x)\,dx$$

$$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx, \text{ if } c \text{ is a constant}$$

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Common Applications of the Definite Integral

Area (Rectangular Coordinates)

Given the function $y = f(x)$ such that $y > 0$ for all x between a and b, the area bounded by the curve $y = f(x)$, the x-axis, and the vertical lines $x = a$ and $x = b$ is

$$A = \int_a^b f(x)\,dx$$

Length of Arc (Rectangular Coordinates)

Given the smooth curve $f(x, y) = 0$ from point (x_1, y_1) to point (x_2, y_2), the length between these points is

$$L = \int_{x_1}^{x_2} \sqrt{1 + (dy/dx)^2}\,dx$$

$$L = \int_{y_1}^{y_2} \sqrt{1 + (dx/dy)^2}\,dy$$

Mean Value of a Function

The mean value of a function $f(x)$ continuous on $[a, b]$ is

$$\frac{1}{(b-a)} \int_a^b f(x)\,dx$$

© 2003 by CRC Press LLC

Area (Polar Coordinates)

Given the curve $r = f(\theta)$, continuous and non-negative for $\theta_1 \le \theta \le \theta_2$, the area enclosed by this curve and the radial lines $\theta = \theta_1$ and $\theta = \theta_2$ is given by

$$A = \int_{\theta_1}^{\theta_2} \frac{1}{2}[f(\theta)]^2 d\theta$$

Length of Arc (Polar Coordinates)

Given the curve $r = f(\theta)$ with continuous derivative $f'(\theta)$ on $\theta_1 \le \theta \le \theta_2$, the length of arc from $\theta = \theta_1$ to $\theta = \theta_2$ is

$$L = \int_{\theta_1}^{\theta_2} \sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\, d\theta$$

Volume of Revolution

Given a function $y = f(x)$, continuous and non-negative on the interval (a, b), when the region bounded by $f(x)$ between a and b is revolved about the x-axis, the volume of revolution is

$$V = \pi \int_a^b [f(x)]^2 dx$$

Surface Area of Revolution
(Revolution about the *x*-axis, between *a* and *b*)

If the portion of the curve $y = f(x)$ between $x = a$ and $x = b$ is revolved about the x-axis, the area A of the surface generated is given by the following:

$$A = \int_a^b 2\pi f(x)\{1 + [f'(x)]^2\}^{1/2} dx$$

Work

If a variable force $f(x)$ is applied to an object in the direction of motion along the x-axis between $x = a$ and $x = b$, the work done is

$$W = \int_a^b f(x)\, dx$$

Cylindrical and Spherical Coordinates

a. Cylindrical coordinates (Figure 31)

$$x = r\cos\theta$$
$$y = r\sin\theta$$

element of volume $dV = r\, dr\, d\theta\, dz$.

b. Spherical coordinates (Figure 32)

$$x = \rho\sin\phi\cos\theta$$
$$y = \rho\sin\phi\sin\theta$$
$$z = \rho\cos\phi$$

element of volume $dV = \rho^2 \sin\phi\, d\rho,\, d\phi\, d\theta$.

© 2003 by CRC Press LLC

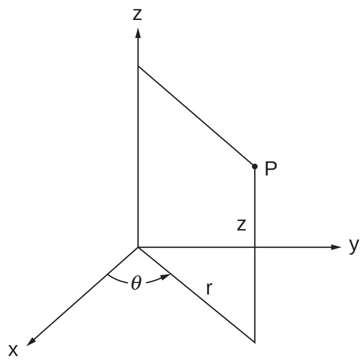


FIGURE 31 Cylindrical coordinates.

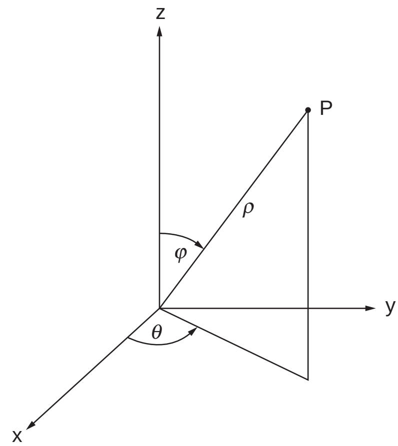


FIGURE 32 Spherical coordinates.

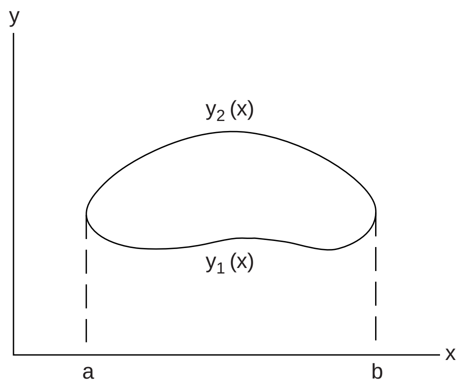


FIGURE 33 Region R bounded by $y_2(x)$ and $y_1(x)$.

Double Integration

The evaluation of a double integral of $f(x, y)$ over a plane region R

$$\iint_R f(x, y)\, dA$$

is practically accomplished by iterated (repeated) integration. For example, suppose that a vertical straight line meets the boundary of R in at most two points so that there is an upper boundary, $y = y_2(x)$, and a lower boundary, $y = y_1(x)$. Also, it is assumed that these functions are continuous from a to b (see Figure 33). Then

$$\iint_R f(x, y)\, dA = \int_a^b \left(\int_{y_1(x)}^{y_2(x)} f(x, y)\, dy \right) dx$$

If R has a left-hand boundary, $x = x_1(y)$, and a right-hand boundary, $x = x_2(y)$, which are continuous from c to d (the extreme values of y in R), then

$$\iint_R f(x, y)\, dA = \int_c^d \left(\int_{x_1(y)}^{x_2(y)} f(x, y)\, dx \right) dy$$

© 2003 by CRC Press LLC

Such integrations are sometimes more convenient in polar coordinates, $x = r\cos\theta$, $y = r\sin\theta$; $dA = r\,dr\,d\theta$.

Surface Area and Volume by Double Integration

For the surface given by $z = f(x, y)$, which projects onto the closed region R of the x–y-plane, one may calculate the volume V bounded above by the surface and below by R, and the surface area S by the following:

$$V = \iint_R z\,dA = \iint_R f(x, y)\,dx\,dy$$

$$S = \iint_R [1 + (\delta z/\delta x)^2 + (\delta z/\delta y)^2]^{1/2}\,dx\,dy$$

[In polar coordinates (r, θ), we replace dA by $r\,dr\,d\theta$].

Centroid

The centroid of a region R of the x–y-plane is a point (x', y') where

$$x' = \frac{1}{A}\iint_R x\,dA \qquad y' = \frac{1}{A}\iint_R y\,dA$$

and A is the area of the region.

Example.

For the circular sector of angle 2α and radius R, the area A is αR^2; the integral needed for x', expressed in polar coordinates, is

$$\iint x\,dA = \int_{-\alpha}^{\alpha}\int_0^R (r\cos\theta)r\,dr\,d\theta$$

$$= \left[\frac{R^3}{3}\sin\theta\right]_{-\alpha}^{+\alpha} = \frac{2}{3}R^3\sin\alpha$$

Thus,

$$x' = \frac{\frac{2}{3}R^3\sin\alpha}{\alpha R^2} = \frac{2}{3}R\frac{\sin\alpha}{\alpha}$$

Centroids of some common regions are shown in Figure 34.

Vector Analysis

Vectors

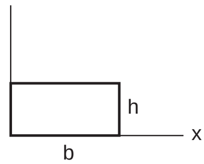
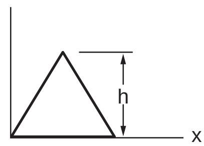
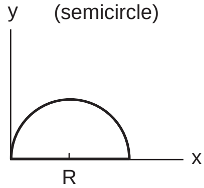
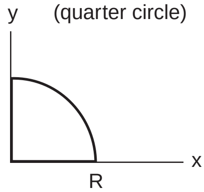
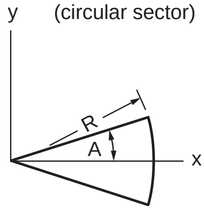
Given the set of mutually perpendicular unit vectors **i, j,** and **k** (Figure 35), any vector in the space may be represented as $\mathbf{F} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$, where a, b, and c are *components*.

Magnitude of F

$$|\mathbf{F}| = (a^2 + b^2 + c^2)^{\frac{1}{2}}$$

© 2003 by CRC Press LLC

Centroids

		Area	*x′*	*y′*
	(rectangle)	bh	b/2	h/2
	(isos. triangle)*	bh/2	b/2	h/3
	(semicircle)	$\pi R^2/2$	R	$4R/3\pi$
	(quarter circle)	$\pi R^2/4$	$4R/3\pi$	$4R/3\pi$
	(circular sector)	R^2A	2R sin A/3A	0

*y′ = h/3 for any triangle of altitude h.

FIGURE 34

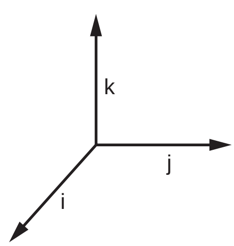


FIGURE 35 The unit vectors **i**, **j**, and **k**.

© 2003 by CRC Press LLC

Product by Scalar p

$$p\mathbf{F} = pa\mathbf{i} + pb\mathbf{j} + pc\mathbf{k}$$

Sum of $\mathbf{F}_1$ and $\mathbf{F}_2$

$$\mathbf{F}_1 + \mathbf{F}_2 = (a_1 + a_2)\mathbf{i} + (b_1 + b_2)\mathbf{j} + (c_1 + c_2)\mathbf{k}$$

Scalar Product

$$\mathbf{F}_1 \cdot \mathbf{F}_2 = a_1 a_2 + b_1 b_2 + c_1 c_2$$

(Thus, $\mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{k} = 1$ and $\mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{k} = \mathbf{k} \cdot \mathbf{i} = \mathbf{0}$.) Also,

$$\mathbf{F}_1 \cdot \mathbf{F}_2 = \mathbf{F}_2 \cdot \mathbf{F}_1$$

$$(\mathbf{F}_1 + \mathbf{F}_2) \cdot \mathbf{F}_3 = \mathbf{F}_1 \cdot \mathbf{F}_3 + \mathbf{F}_2 \cdot \mathbf{F}_3$$

Vector Product

$$\mathbf{F}_1 \times \mathbf{F}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$$

(Thus, $\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = \mathbf{0}$, $\mathbf{i} \times \mathbf{j} = \mathbf{k}$, $\mathbf{j} \times \mathbf{k} = \mathbf{i}$, and $\mathbf{k} \times \mathbf{i} = \mathbf{j}$.) Also,

$$\begin{aligned} \mathbf{F}_1 \times \mathbf{F}_2 &= -\mathbf{F}_2 \times \mathbf{F}_1 \\ (\mathbf{F}_1 + \mathbf{F}_2) \times \mathbf{F}_3 &= \mathbf{F}_1 \times \mathbf{F}_3 + \mathbf{F}_2 \times \mathbf{F}_3 \\ \mathbf{F}_1 \times (\mathbf{F}_2 + \mathbf{F}_3) &= \mathbf{F}_1 \times \mathbf{F}_2 + \mathbf{F}_1 \times \mathbf{F}_3 \\ \mathbf{F}_1 \times (\mathbf{F}_2 \times \mathbf{F}_3) &= (\mathbf{F}_1 \cdot \mathbf{F}_3)\mathbf{F}_2 - (\mathbf{F}_1 \cdot \mathbf{F}_2)\mathbf{F}_3 \\ \mathbf{F}_1 \cdot (\mathbf{F}_2 \times \mathbf{F}_3) &= (\mathbf{F}_1 \times \mathbf{F}_2) \cdot \mathbf{F}_3 \end{aligned}$$

Vector Differentiation

If $\mathbf{V}$ is a vector function of a scalar variable t, then

$$\mathbf{V} = a(t)\mathbf{i} + b(t)\mathbf{j} + c(t)\mathbf{k}$$

and

$$\frac{d\mathbf{V}}{dt} = \frac{da}{dt}\mathbf{i} + \frac{db}{dt}\mathbf{j} + \frac{dc}{dt}\mathbf{k}$$

For several vector functions $\mathbf{V}_1, \mathbf{V}_2, \ldots, \mathbf{V}_n$

$$\begin{aligned} \frac{d}{dt}(\mathbf{V}_1 + \mathbf{V}_2 + \cdots + \mathbf{V}_n) &= \frac{d\mathbf{V}_1}{dt} + \frac{d\mathbf{V}_2}{dt} + \cdots + \frac{d\mathbf{V}_n}{dt} \\ \frac{d}{dt}(\mathbf{V}_1 \cdot \mathbf{V}_2) &= \frac{d\mathbf{V}_1}{dt} \cdot \mathbf{V}_2 + \mathbf{V}_1 \cdot \frac{d\mathbf{V}_2}{dt} \\ \frac{d}{dt}(\mathbf{V}_1 \times \mathbf{V}_2) &= \frac{d\mathbf{V}_1}{dt} \times \mathbf{V}_2 + \mathbf{V}_1 \times \frac{d\mathbf{V}_2}{dt} \end{aligned}$$

© 2003 by CRC Press LLC

For a scalar-valued function $g(x, y, z)$

$$\textbf{(gradient)} \quad \text{grad } g = \nabla g = \frac{\delta g}{\delta x}\mathbf{i} + \frac{\delta g}{\delta y}\mathbf{j} + \frac{\delta g}{\delta z}\mathbf{k}$$

For a vector-valued function $\mathbf{V}(a, b, c)$, where a, b, and c are each a function of x, y, and z,

(divergence)

$$\text{div}\mathbf{V} = \nabla \cdot \mathbf{V} = \frac{\delta a}{\delta x} + \frac{\delta b}{\delta y} + \frac{\delta c}{\delta z}$$

(**curl**)

$$\text{curl}\mathbf{V} = \nabla \times \mathbf{V} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\delta}{\delta x} & \dfrac{\delta}{\delta y} & \dfrac{\delta}{\delta z} \\ a & b & c \end{vmatrix}$$

Also,

$$\text{div grad } g = \nabla^2 g = \frac{\delta^2 g}{\delta x^2} + \frac{\delta^2 g}{\delta y^2} + \frac{\delta^2 g}{\delta z^2}$$

and

$$\text{curl grad } g = \mathbf{0}; \qquad \text{div curl } \mathbf{V} = 0;$$

$$\text{curl curl}\mathbf{V} = \text{grad div}\mathbf{V} - (\mathbf{i}\nabla^2 a + \mathbf{j}\nabla^2 b + \mathbf{k}\nabla^2 c)$$

Divergence Theorem (Gauss)

Given a vector function $\boldsymbol{F}$ with continuous partial derivatives in a region R bounded by a closed surface S, then

$$\iiint_R \text{div} \cdot \mathbf{F}\, dV = \iint_S \mathbf{n} \cdot \mathbf{F}\, dS$$

where $\mathbf{n}$ is the (sectionally continuous) unit normal to S.

Stokes' Theorem

Given a vector function with continuous gradient over a surface S that consists of portions that are piecewise smooth and bounded by regular closed curves such as C,

$$\iint_S \mathbf{n} \cdot \text{curl } \mathbf{F} dS = \oint_C \mathbf{F} \cdot d\mathbf{r}$$

Planar Motion in Polar Coordinates

Motion in a plane may be expressed with regard to polar coordinates (r, θ). Denoting the position vector by $\mathbf{r}$ and its magnitude by r, we have $\mathbf{r} = r\mathbf{R}(\theta)$, where $\mathbf{R}$ is the unit vector. Also, $d\mathbf{R}/d\theta = \mathbf{P}$, a unit vector perpendicular to $\mathbf{R}$. The velocity and acceleration are then

$$\mathbf{v} = \frac{dr}{dt}\mathbf{R} + r\frac{d\theta}{dt}\mathbf{P}$$

$$\mathbf{a} = \left[\frac{d^2 r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right]\mathbf{R} + \left[r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}\right]\mathbf{P}$$

© 2003 by CRC Press LLC

Note that the component of acceleration in the **P** direction (transverse component) may also be written

$$\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)$$

so that in purely radial motion it is zero and

$$r^2\frac{d\theta}{dt} = C\ (\text{constant})$$

which means that the position vector sweeps out area at a constant rate [see Area (Polar Coordinates) in the section entitled Integral Calculus].

Special Functions

Hyperbolic Functions

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \operatorname{csch} x = \frac{1}{\sinh x}$$

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \operatorname{sech} x = \frac{1}{\cosh x}$$

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} \qquad \operatorname{ctnh} x = \frac{1}{\tanh x}$$

$$\sinh(-x) = -\sinh x \qquad \operatorname{ctnh}(-x) = -\operatorname{ctnh} x$$

$$\cosh(-x) = \cosh x \qquad \operatorname{sech}(-x) = \operatorname{sech} x$$

$$\tanh(-x) = -\tanh x \qquad \operatorname{csch}(-x) = -\operatorname{csch} x$$

$$\tanh x = \frac{\sinh x}{\cosh x} \qquad \operatorname{ctnh} x = \frac{\cosh x}{\sinh x}$$

$$\cosh^2 x - \sinh^2 x = 1 \qquad \cosh^2 x = \frac{1}{2}(\cosh 2x + 1)$$

$$\sinh^2 x = \frac{1}{2}(\cosh 2x - 1) \qquad \operatorname{ctnh}^2 x - \operatorname{csch}^2 x = 1$$

$$\operatorname{csch}^2 x - \operatorname{sech}^2 x = \operatorname{csch}^2 x \operatorname{sech}^2 x \qquad \tanh^2 x + \operatorname{sech}^2 x = 1$$

$$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$$

$$\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$$

$$\sinh(x-y) = \sinh x \cosh y - \cosh x \sinh y$$

$$\cosh(x-y) = \cosh x \cosh y - \sinh x \sinh y$$

$$\tanh(x+y) = \frac{\tanh x + \tanh y}{1 + \tanh x \tanh y}$$

$$\tanh(x-y) = \frac{\tanh x - \tanh y}{1 - \tanh x \tanh y}$$

Laplace Transforms

The Laplace transform of the function $f(t)$, denoted by $F(s)$ or $L\{f(t)\}$, is defined

$$F(s) = \int_0^\infty f(t)e^{-st}dt$$

© 2003 by CRC Press LLC

provided that the integration may be validly performed. A sufficient condition for the existence of $F(s)$ is that $f(t)$ be of exponential order as $t \to \infty$ and that it is sectionally continuous over every finite interval in the range $t \geq 0$. The Laplace transform of $g(t)$ is denoted by $L\{g(t)\}$ or $G(s)$.

Operations

$f(t)$	$F(s) = \int_0^\infty f(t)e^{-st}\,dt$
$af(t) + bg(t)$	$aF(s) + bG(s)$
$f'(t)$	$sF(s) - f(0)$
$f''(t)$	$s^2F(s) - sf(0) - f'(0)$
$f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0)$
$tf(t)$	$-F'(s)$
$t^nf(t)$	$(-1)^nF^{(n)}(s)$
$e^{at}f(t)$	$F(s-a)$
$\int_0^t f(t-\beta)\cdot g(\beta)d\beta$	$F(s)\cdot G(s)$
$f(t-a)$	$e^{-as}F(s)$
$f\left(\frac{t}{a}\right)$	$aF(as)$
$\int_0^t g(\beta)\,d\beta$	$\frac{1}{s}G(s)$
$f(t-c)\delta(t-c)$	$e^{-cs}F(s),\ c > 0$

where

$$\begin{aligned}\delta(t-c) &= 0 \text{ if } 0 \leq t < c \\ &= 1 \text{ if } t \geq c\end{aligned}$$

$f(t) = f(t+\omega)$ (periodic)	$\dfrac{\int_0^\omega e^{-s\tau}f(\tau)d\tau}{1 - e^{-s\omega}}$

Table of Laplace Transforms

$f(t)$	$F(s)$	$f(t)$	$F(s)$	
1	$1/s$	$\sinh at$	$\dfrac{a}{s^2 - a^2}$	
t	$1/s^2$	$\cosh at$	$\dfrac{s}{s^2 - a^2}$	
$\dfrac{t^{n-1}}{(n-1)!}$	$1/s^n\ (n = 1, 2, 3, \ldots)$	$e^{at} - e^{bt}$	$\dfrac{a-b}{(s-a)(s-b)}$	$(a \neq b)$
$\sqrt{t}$	$\dfrac{1}{2s}\sqrt{\dfrac{\pi}{s}}$	$ae^{at} - be^{bt}$	$\dfrac{s(a-b)}{(s-a)(s-b)}$	$(a \neq b)$
$\dfrac{1}{\sqrt{t}}$	$\sqrt{\dfrac{\pi}{s}}$	$t \sin at$	$\dfrac{2as}{(s^2+a^2)^2}$	

© 2003 by CRC Press LLC

e^{at}	$\dfrac{1}{s-a}$	$t \cos at$	$\dfrac{s^2 - a^2}{(s^2 + a^2)^2}$
te^{at}	$\dfrac{1}{(s-a)^2}$	$e^{at} \sin bt$	$\dfrac{b}{(s-a)^2 + b^2}$
$\dfrac{t^{n-1}e^{at}}{(n-1)!}$	$\dfrac{1}{(s-a)^n}$ $(n = 1, 2, 3, \ldots)$	$e^{at} \cos bt$	$\dfrac{s-a}{(s-a)^2 + b^2}$
$\dfrac{t^x}{\Gamma(x+1)}$	$\dfrac{1}{s^{x+1}}$, $x > -1$	$\dfrac{\sin at}{t}$	$\text{Arc} \tan \dfrac{a}{s}$
$\sin at$	$\dfrac{a}{s^2 + a^2}$	$\dfrac{\sinh at}{t}$	$\dfrac{1}{2} \log_e \left(\dfrac{s+a}{s-a} \right)$
$\cos at$	$\dfrac{s}{s^2 + a^2}$		

z-Transform

For the real-valued sequence $\{f(k)\}$ and complex variable z, the z-transform, $F(z) = Z\{f(k)\}$, is defined by

$$Z\{f(k)\} = F(z) = \sum_{k=0}^{\infty} f(k) z^{-k}$$

For example, the sequence $f(k) = 1$, $k = 0, 1, 2, \ldots$, has the z-transform

$$F(z) = 1 + z^{-1} + z^{-2} + z^{-3} \cdots + z^{-k} + \cdots$$

Angles are measured in degrees or radians: $180^\circ = \pi$ radians; 1 radian $= 180^\circ/\pi$ degrees.
The trigonometric functions of 0°, 30°, 45°, and integer multiples of these are directly computed.

	0°	30°	45°	60°	90°	120°	135°	150°	180°
sin	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1
tan	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	∞	$-\sqrt{3}$	-1	$-\frac{\sqrt{3}}{3}$	0
ctn	∞	$\sqrt{3}$	1	$\frac{\sqrt{3}}{3}$	0	$-\frac{\sqrt{3}}{3}$	-1	$-\sqrt{3}$	∞
sec	1	$\frac{2\sqrt{3}}{3}$	$\sqrt{2}$	2	∞	-2	$-\sqrt{2}$	$-\frac{2\sqrt{3}}{3}$	-1
csc	∞	2	$\sqrt{2}$	$\frac{2\sqrt{3}}{3}$	1	$\frac{2\sqrt{3}}{3}$	$\sqrt{2}$	2	∞

Trigonometric Identities

$$\sin A = \frac{1}{\csc A}$$

$$\cos A = \frac{1}{\sec A}$$

© 2003 by CRC Press LLC

Defining $z = e^{sT}$ gives

$$\mathrm{L}\{U^*(t)\} = \sum_{k=0}^{\infty} U(kT)z^{-k}$$

which is the z-transform of the sampled signal $U(kT)$.

Properties

Linearity: $Z\{af_1(k) + bf_2(k)\} = aZ\{f_1(k)\} + bZ\{f_2(k)\} = aF_1(z) + bF_2(z)$

Right-shifting property: $Z\{f(k-n)\} = z^{-n}F(z)$

Left-shifting property: $Z\{f(k+n)\} = z^n F(z) - \sum_{k=0}^{n-1} f(k)z^{n-k}$

Time scaling: $Z\{a^k f(k)\} = F(z/a)$

Multiplication by k: $Z\{kf(k)\} = -z dF(z)/dz$

Initial value: $f(0) = \lim_{z\to\infty}(1-z^{-1})F(z) = F(\infty)$

Final value: $\lim_{k\to\infty} f(k) = \lim_{z\to 1}(1-z^{-1})F(z)$

Convolution: $Z\{f_1(k)^* f_2(k)\} = F_1(z)F_2(z)$

z-Transforms of Sampled Functions

$f(k)$	$Z\{f(kT)\} = F(z)$
1 at k; else 0	z^{-k}
1	$\dfrac{z}{z-1}$
kT	$\dfrac{Tz}{(z-1)^2}$
$(kT)^2$	$\dfrac{T^2 z(z+1)}{(z-1)^3}$
$\sin \omega kT$	$\dfrac{z \sin \omega T}{z^2 - 2z \cos \omega T + 1}$
$\cos \omega T$	$\dfrac{z(z - \cos \omega T)}{z^2 - 2z \cos \omega T + 1}$
e^{-akT}	$\dfrac{z}{z - e^{-aT}}$
kTe^{-akT}	$\dfrac{zTe^{-aT}}{(z - e^{-aT})^2}$
$(kT)^2 e^{-akT}$	$\dfrac{T^2 e^{-aT} z(z + e^{-aT})}{(z - e^{-aT})^3}$
$e^{-akT} \sin \omega kT$	$\dfrac{ze^{-aT} \sin \omega T}{z^2 - 2ze^{-aT} \cos \omega T + e^{-2aT}}$
$e^{-akT} \cos \omega kT$	$\dfrac{z(z - e^{-aT} \cos \omega T)}{z^2 - 2ze^{-aT} \cos \omega T + e^{-2aT}}$
$a^k \sin \omega kT$	$\dfrac{az \sin \omega T}{z^2 - 2az \cos \omega T + a^2}$
$a^k \cos \omega kT$	$\dfrac{z(z - a \cos \omega T)}{z^2 - 2az \cos \omega T + a^2}$

© 2003 by CRC Press LLC

Fourier Series

The periodic function $f(t)$ with period 2π may be represented by the trigonometric series

$$a_0 + \sum_1^\infty (a_n \cos nt + b_n \sin nt)$$

where the coefficients are determined from

$$\begin{aligned} a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,dt \\ a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt\,dt \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin nt\,dt \qquad (n = 1, 2, 3, \ldots) \end{aligned}$$

Such a trigonometric series is called the Fourier series corresponding to $f(t)$ and the coefficients are termed Fourier coefficients of $f(t)$. If the function is piecewise continuous in the interval $-\pi \le t \le \pi$ and has left- and right-hand derivatives at each point in that interval, then the series is convergent with sum $f(t)$ except at points t_i, at which $f(t)$ is discontinuous. At such points of discontinuity, the sum of the series is the arithmetic mean of the right- and left-hand limits of $f(t)$ at t_i. The integrals in the formulas for the Fourier coefficients can have limits of integration that span a length of 2π, for example, 0 to 2π (because of the periodicity of the integrands).

Functions with Period Other Than 2π

If $f(t)$ has period P, the Fourier series is

$$f(t) \sim a_0 + \sum_1^\infty \left(a_n \cos\frac{2\pi n}{P}t + b_n \sin\frac{2\pi n}{P}t\right)$$

where

$$\begin{aligned} a_0 &= \frac{1}{P}\int_{-P/2}^{P/2} f(t)\,dt \\ a_n &= \frac{2}{P}\int_{-P/2}^{P/2} f(t)\cos\frac{2\pi n}{P}t\,dt \\ b_n &= \frac{2}{P}\int_{-P/2}^{P/2} f(t)\sin\frac{2\pi n}{P}t\,dt \end{aligned}$$

Again, the interval of integration in these formulas may be replaced by an interval of length P, for example, 0 to P.

Bessel Functions

Bessel functions, also called cylindrical functions, arise in many physical problems as solutions of the differential equation

$$x^2y'' + xy' + (x^2 - n^2)y = 0$$

© 2003 by CRC Press LLC

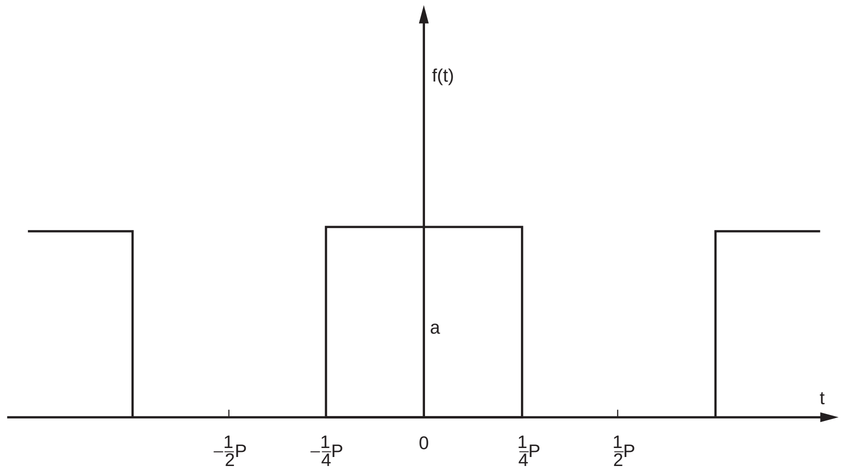


FIGURE 36 Square wave. $f(t) \sim \frac{a}{2} + \frac{2a}{\pi}\left(\cos\frac{2\pi t}{P} - \frac{1}{3}\cos\frac{6\pi t}{P} + \frac{1}{3}\cos\frac{10\pi t}{P} + \cdots\right).$

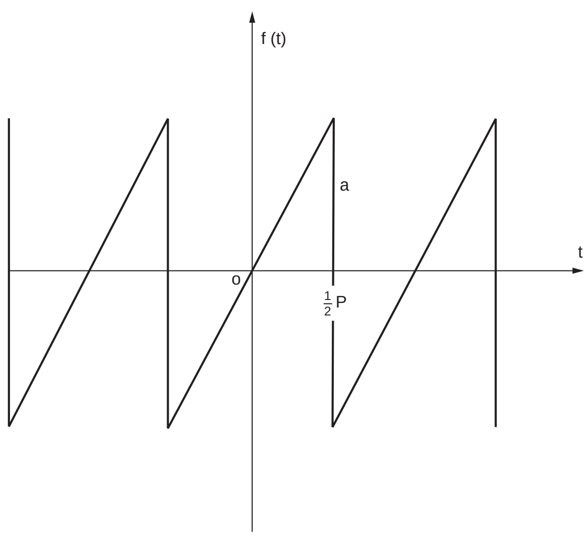


FIGURE 37 Sawtooth wave. $f(t) \sim \frac{2a}{\pi}\left(\sin\frac{2\pi t}{P} - \frac{1}{2}\sin\frac{4\pi t}{P} + \frac{1}{3}\sin\frac{6\pi t}{P} - \cdots\right).$

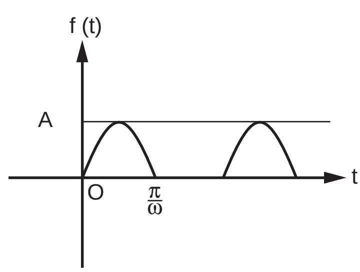


FIGURE 38 Half-wave rectifier. $f(t) \sim \frac{A}{\pi} + \frac{A}{2}\sin\omega t - \frac{2A}{\pi}\left(\frac{1}{(1)(3)}\cos 2\omega t + \frac{1}{(3)(5)}\cos 4\omega t + \cdots\right).$

© 2003 by CRC Press LLC

which is known as Bessel's equation. Certain solutions of the above, known as *Bessel functions of the first kind of order n,* are given by

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(n+k+1)}\left(\frac{x}{2}\right)^{n+2k}$$

$$J_{-n}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(-n+k+1)}\left(\frac{x}{2}\right)^{-n+2k}$$

In the above it is noteworthy that the gamma function must be defined for the negative argument q: $\Gamma(q) = \Gamma(q+1)/q$, provided that q is not a negative integer. When q is a negative integer, $1/\Gamma(q)$ is defined to be zero. The functions $J_{-n}(x)$ and $J_n(x)$ are solutions of Bessel's equation for all real n. It is seen, for $n = 1, 2, 3, \ldots$, that

$$J_{-n}(x) = (-1)^n J_n(x)$$

and, therefore, these are not independent; hence, a linear combination of these is not a general solution. When, however, n is not a positive integer, a negative integer, or zero, the linear combination with arbitrary constants c_1 and c_2

$$y = c_1 J_n(x) + c_2 J_{-n}(x)$$

is the general solution of the Bessel differential equation.

The zero-order function is especially important as it arises in the solution of the heat equation (for a "long" cylinder):

$$J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 4^2} - \frac{x^6}{2^2 4^2 6^2} + \cdots$$

while the following relations show a connection to the trigonometric functions:

$$J_{\frac{1}{2}}(x) = \left[\frac{2}{\pi x}\right]^{1/2} \sin x$$

$$J_{-\frac{1}{2}}(x) = \left[\frac{2}{\pi x}\right]^{1/2} \cos x$$

The following recursion formula gives $J_{n+1}(x)$ for any order in terms of lower-order functions:

$$\frac{2n}{x} J_n(x) = J_{n-1}(x) + J_{n+1}(x)$$

Legendre Polynomials

If Laplace's equation, $\nabla^2 V = 0$, is expressed in spherical coordinates, it is

$$r^2 \sin\theta \frac{\delta^2 V}{\delta r^2} + 2r\sin\theta \frac{\delta V}{\delta r} + \sin\theta \frac{\delta^2 V}{\delta\theta^2} + \cos\theta \frac{\delta V}{\delta\theta} + \frac{1}{\sin\theta} \frac{\delta^2 V}{\delta\phi^2} = 0$$

and any of its solutions, $V(r, \theta, \phi)$, are known as *spherical harmonics.* The solution as a product

© 2003 by CRC Press LLC

$$V(r, \theta, \phi) = R(r)\Theta(\theta)$$

which is independent of ϕ, leads to

$$\sin^2\theta\Theta'' + \sin\theta\cos\theta\Theta' + [n(n+1)\sin^2\theta]\Theta = 0$$

Rearrangement and substitution of $x = \cos\theta$ leads to

$$(1-x^2)\frac{d^2\Theta}{dx^2} - 2x\frac{d\Theta}{dx} + n(n+1)\Theta = 0$$

known as *Legendre's equation*. Important special cases are those in which n is zero or a positive integer, and, for such cases, Legendre's equation is satisfied by polynomials called Legendre polynomials, $P_n(x)$. A short list of Legendre polynomials, expressed in terms of x and $\cos\theta$, is given below. These are given by the following general formula:

$$P_n(x) = \sum_{j=0}^{L} \frac{(-1)^j(2n-2j)!}{2^n j!(n-j)!(n-2j)!} x^{n-2j}$$

where $L = n/2$ if n is even and $L = (n-1)/2$ if n is odd.

$$P_0(x) = 1$$
$$P_1(x) = x$$
$$P_2(x) = \frac{1}{2}(3x^2 - 1)$$
$$P_3(x) = \frac{1}{2}(5x^3 - 3x)$$
$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$$
$$P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)$$
$$P_0(\cos\theta) = 1$$
$$P_1(\cos\theta) = \cos\theta$$
$$P_2(\cos\theta) = \frac{1}{4}(3\cos 2\theta + 1)$$
$$P_3(\cos\theta) = \frac{1}{8}(5\cos 3\theta + 3\cos\theta)$$
$$P_4(\cos\theta) = \frac{1}{64}(35\cos 4\theta + 20\cos 2\theta + 9)$$

Additional Legendre polynomials may be determined from the *recursion formula*

$$(n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x) = 0 \qquad (n = 1, 2, \ldots)$$

or the *Rodrigues formula*

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$$

© 2003 by CRC Press LLC

Laguerre Polynomials

Laguerre polynomials, denoted $L_n(x)$, are solutions of the differential equation

$$xy'' + (1-x)y' + ny = 0$$

and are given by

$$L_n(x) = \sum_{j=0}^{n} \frac{(-1)^j}{j!} C_{(n,j)} x^j \qquad (n = 0, 1, 2, \ldots)$$

Thus,

$$L_0(x) = 1$$
$$L_1(x) = 1 - x$$
$$L_2(x) = 1 - 2x + \frac{1}{2}x^2$$
$$L_3(x) = 1 - 3x + \frac{3}{2}x^2 - \frac{1}{6}x^3$$

Additional Laguerre polynomials may be obtained from the recursion formula

$$(n+1)L_{n+1}(x) - (2n+1-x)L_n(x) + nL_{n-1}(x) = 0$$

Hermite Polynomials

The Hermite polynomials, denoted $H_n(x)$, are given by

$$H_0 = 1, \qquad H_n(x) = (-1)^n e^{x^2} \frac{d^n e^{-x^2}}{dx^n} \qquad (n = 1, 2, \ldots)$$

and are solutions of the differential equation

$$y'' - 2xy' + 2ny = 0 \qquad (n = 0, 1, 2, \ldots)$$

The first few Hermite polynomials are

$$H_0 = 1 \qquad H_1(x) = 2x$$
$$H_2(x) = 4x^2 - 2 \qquad H_3(x) = 8x^2 - 12x$$
$$H_4(x) = 16x^4 - 48x^2 + 12$$

Additional Hermite polynomials may be obtained from the relation

$$H_{n+1}(x) = 2xH_n(x) - H'_n(x)$$

where prime denotes differentiation with respect to x.

Orthogonality

A set of functions $\{f_n(x)\}$ $(n = 1, 2, \ldots)$ is orthogonal in an interval (a, b) with respect to a given weight function $w(x)$ if

$$\int_a^b w(x) f_m(x) f_n(x)\, dx = 0 \qquad \text{when } m \neq n$$

© 2003 by CRC Press LLC

The following polynomials are orthogonal on the given interval for the given $w(x)$:

Legendre polynomials:	$P_n(x)$	$w(x) = 1$
		$a = -1, b = 1$
Laguerre polynomials:	$L_n(x)$	$w(x) = \exp(-x)$
		$a = 0, b = \infty$
Hermite polynomials	$H_n(x)$	$w(x) = \exp(-x^2)$
		$a = -\infty, b = \infty$

The Bessel functions *of order n*, $J_n(\lambda_1 x)$, $J_n(\lambda_2 x)$, …, are orthogonal with respect to $w(x) = x$ over the interval $(0, c)$, provided that the λ_i are the positive roots of $J_n(\lambda c) = 0$:

$$\int_0^c x J_n(\lambda_j x) J_n(\lambda_k x)\, dx = 0 \qquad (j \neq k)$$

where n is fixed and $n \geq 0$.

Statistics

Arithmetic Mean

$$\mu = \frac{\Sigma X_i}{N}$$

where X_i is a measurement in the population and N is the total number of X_i in the population. For a *sample* of size n, the sample mean, denoted $\bar{X}$, is

$$\bar{X} = \frac{\Sigma X_i}{n}$$

Median

The median is the middle measurement when an odd number (n) of measurements is arranged in order; if n is even, it is the midpoint between the two middle measurements.

Mode

The mode is the most frequently occurring measurement in a set.

Geometric Mean

$$\text{geometric mean} = \sqrt[n]{X_1 X_2 \ldots X_n}$$

Harmonic Mean

The harmonic mean H of n numbers $X_1, X_2, \ldots, X_n$ is

$$H = \frac{n}{\Sigma(1/(Xi))}$$

© 2003 by CRC Press LLC

Variance

The mean of the sum of squares of deviations from the mean (μ) is the population variance, denoted σ^2:

$$\sigma^2 = \Sigma(X_i - \mu)^2 / N$$

The sample variance, s^2, for sample size n is

$$s^2 = \Sigma(X_i - \bar{X})^2 / (n-1)$$

A simpler computational form is

$$s^2 = \frac{\Sigma X_i^2 - \frac{(\Sigma X_i)^2}{n}}{n-1}$$

Standard Deviation

The positive square root of the population variance is the standard deviation. For a population,

$$\sigma = \left[\frac{\Sigma X_i^2 - \frac{(\Sigma X_i)^2}{N}}{N}\right]^{1/2}$$

for a sample

$$s = \left[\frac{\Sigma X_i^2 - \frac{(\Sigma X_i)^2}{n}}{n-1}\right]^{1/2}$$

Coefficient of Variation

$$V = s / \bar{X}$$

Probability

For the sample space U, with subsets A of U (called "events"), we consider the probability measure of an event A to be a real-valued function p defined over all subsets of U such that:

$$0 \le p(A) \le 1$$
$$p(U) = 1 \text{ and } p(\Phi) = 0$$

If A_1 and A_2 are subsets of U, then

$$p(A_1 \cup A_2) = p(A_1) + p(A_2) - p(A_1 \cap A_2)$$

Two events A_1 and A_2 are called mutually exclusive if and only if $A_1 \cap A_2 = \phi$ (null set). These events are said to be independent if and only if $p(A_1 \cap A_2) = p(A_1)p(A_2)$.

© 2003 by CRC Press LLC

Conditional Probability and Bayes' Rule

The probability of an event A, given that an event B has occurred, is called the conditional probability and is denoted $p(A/B)$. Further,

$$p(A/B) = \frac{p(A \cap B)}{p(B)}$$

Bayes' rule permits a calculation of *a posteriori* probability from given *a priori* probabilities and is stated below:

If $A_1, A_2, \ldots, A_n$ are n mutually exclusive events, and $p(A_1) + p(A_2) + \ldots + p(A_n) = 1$, and B is any event such that $p(B)$ is not 0, then the conditional probability $p(A_i/B)$ for any one of the events A_i, *given that B has occurred*, is

$$p(A_i/B) = \frac{P(A_i)p(B/A_i)}{p(A_1)p(B/A_1) + p(A_2)p(B/A_2) + \cdots + p(A_n)p(B/A_n)}$$

Example

Among five different laboratory tests for detecting a certain disease, one is effective with probability 0.75, whereas each of the others is effective with probability 0.40. A medical student, unfamiliar with the advantage of the best test, selects one of them and is successful in detecting the disease in a patient. What is the probability that the most effective test was used?

Let B denote (the event) of detecting the disease, A_1 the selection of the best test, and A_2 the selection of one of the other four tests; thus, $p(A_1) = 1/5$, $p(A_2) = 4/5$, $p(B/A_1) = 0.75$, and $p(B/A_2) = 0.40$. Therefore,

$$p(A_1/B) = \frac{\frac{1}{5}(0.75)}{\frac{1}{5}(0.75) + \frac{4}{5}(0.40)} = 0.319$$

Note that the *a priori* probability is 0.20; the outcome raises this probability to 0.319.

Binomial Distribution

In an experiment consisting of n independent trials in which an event has probability p in a single trial, the probability P_X of obtaining X successes is given by

$$P_X = C_{(n,X)} p^X q^{(n-X)}$$

where

$$q = (1-p) \text{ and } C_{(n,X)} = \frac{n!}{X!(n-X)!}$$

The probability of between a and b successes (both a and b included) is $P_a + P_{a+1} + \cdots + P_b$, so if $a = 0$ and $b = n$, this sum is

$$\sum_{X=0}^{n} C_{(n,X)} p^X q^{(n-X)} = q^n + C_{(n,1)} q^{n-1} p + C_{(n,2)} q^{n-2} p^2 + \cdots + p^n = (q+p)^n = 1$$

Mean of Binomially Distributed Variable

The mean number of successes in n independent trials is $m = np$, with standard deviation $\sigma = \sqrt{npq}$.

© 2003 by CRC Press LLC

Normal Distribution

In the binomial distribution, as n increases, the histogram of heights is approximated by the bell-shaped curve (normal curve)

$$Y = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-m)^2/2\sigma^2}$$

where m = the mean of the binomial distribution = np, and $\sigma = \sqrt{npq}$ is the standard deviation. For any normally distributed random variable X with mean m and standard deviation σ, the probability function (density) is given by the above.

The *standard* normal probability curve is given by

$$y = \frac{1}{\sqrt{2\pi}} e^{-Z^2/2}$$

and has mean = 0 and standard deviation = 1. The total area under the standard normal curve is 1. Any normal variable X can be put into standard form by defining $Z = (X - m)/\sigma$; thus, the probability of X between a given X_1 and X_2 is the area under the standard normal curve between the corresponding Z_1 and Z_2 (Table 1 in the Tables of Probability and Statistics). The standard normal curve is often used instead of the binomial distribution in experiments with discrete outcomes. For example, to determine the probability of obtaining 60 to 70 heads in a toss of 100 coins, we take $X = 59.5$ to $X = 70.5$ and compute corresponding values of Z from mean $np = 100\ \frac{1}{2} = 50$, and the standard deviation $\sigma = \sqrt{(100)(1/2)(1/2)} = 5$. Thus, $Z = (59.5 - 50)/5 = 1.9$ and $Z = (70.5 - 50)/5 = 4.1$. From Table 1, the area between $Z = 0$ and $Z = 4.1$ is 0.5000 and between $Z = 0$ and $Z = 1.9$ is 0.4713; hence, the desired probability is 0.0287. The binomial distribution requires a more lengthy computation.

$$C_{(100,\,60)}(1/2)^{60}(1/2)^{40} + C_{(100,\,61)}(1/2)^{61}(1/2)^{39} + \cdots + C_{(100,\,70)}(1/2)^{70}(1/2)^{30}$$

Note that the normal curve is symmetric, whereas the histogram of the binomial distribution is symmetric only if $p = q = 1/2$. Accordingly, when p (hence, q) differs appreciably from 1/2, the difference between probabilities computed by each increases. It is usually recommended that the normal approximation not be used if p (or q) is so small that np (or nq) is less than 5.

Poisson Distribution

$$P = \frac{e^{-m} m^r}{r!}$$

is an approximation to the binomial probability for r successes in n trials when $m = np$ is small (< 5) and the normal curve is not recommended to approximate binomial probabilities (Table 2 in the Tables of Probability and Statistics). The variance σ^2 in the Poisson distribution is np, the same value as the mean.

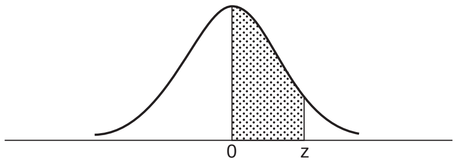
Example

A school's expulsion rate is 5 students per 1000. If class size is 400, what is the probability that 3 or more will be expelled? Since $p = 0.005$ and $n = 400$, $m = np = 2$ and $r = 3$. From Table 2 we obtain for $m = 2$ and $r\ (= x) = 3$ the probability $p = 0.323$.

© 2003 by CRC Press LLC

Tables of Probability and Statistics

TABLE 1 Areas Under the Standard Normal Curve



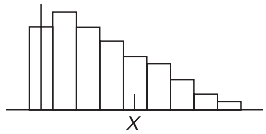
z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.0000	0.0040	0.0080	0.0120	0.0160	0.0199	0.0239	0.0279	0.0319	0.0359
0.1	0.0398	0.0438	0.0478	0.0517	0.0557	0.0596	0.0636	0.0675	0.0714	0.0753
0.2	0.0793	0.0832	0.0871	0.0910	0.0948	0.0987	0.1026	0.1064	0.1103	0.1141
0.3	0.1179	0.1217	0.1255	0.1293	0.1331	0.1368	0.1406	0.1443	0.1480	0.1517
0.4	0.1554	0.1591	0.1628	0.1664	0.1700	0.1736	0.1772	0.1808	0.1844	0.1879
0.5	0.1915	0.1950	0.1985	0.2019	0.2054	0.2088	0.2123	0.2157	0.2190	0.2224
0.6	0.2257	0.2291	0.2324	0.2357	0.2389	0.2422	0.2454	0.2486	0.2517	0.2549
0.7	0.2580	0.2611	0.2642	0.2673	0.2704	0.2734	0.2764	0.2794	0.2823	0.2852
0.8	0.2881	0.2910	0.2939	0.2967	0.2995	0.3023	0.3051	0.3078	0.3106	0.3133
0.9	0.3159	0.3186	0.3212	0.3238	0.3264	0.3289	0.3315	0.3340	0.3365	0.3389
1.0	0.3413	0.3438	0.3461	0.3485	0.3508	0.3531	0.3554	0.3577	0.3599	0.3621
1.1	0.3643	0.3665	0.3686	0.3708	0.3729	0.3749	0.3770	0.3790	0.3810	0.3830
1.2	0.3849	0.3869	0.3888	0.3907	0.3925	0.3944	0.3962	0.3980	0.3997	0.4015
1.3	0.4032	0.4049	0.4066	0.4082	0.4099	0.4115	0.4131	0.4147	0.4162	0.4177
1.4	0.4192	0.4207	0.4222	0.4236	0.4251	0.4265	0.4279	0.4292	0.4306	0.4319
1.5	0.4332	0.4345	0.4357	0.4370	0.4382	0.4394	0.4406	0.4418	0.4429	0.4441
1.6	0.4452	0.4463	0.4474	0.4484	0.4495	0.4505	0.4515	0.4525	0.4535	0.4545
1.7	0.4554	0.4564	0.4573	0.4582	0.4591	0.4599	0.4608	0.4616	0.4625	0.4633
1.8	0.4641	0.4649	0.4656	0.4664	0.4671	0.4678	0.4686	0.4693	0.4699	0.4706
1.9	0.4713	0.4719	0.4726	0.4732	0.4738	0.4744	0.4750	0.4756	0.4761	0.4767
2.0	0.4772	0.4778	0.4783	0.4788	0.4793	0.4798	0.4803	0.4808	0.4812	0.4817
2.1	0.4821	0.4826	0.4830	0.4834	0.4838	0.4842	0.4846	0.4850	0.4854	0.4857
2.2	0.4861	0.4864	0.4868	0.4871	0.4875	0.4878	0.4881	0.4884	0.4887	0.4890
2.3	0.4893	0.4896	0.4898	0.4901	0.4904	0.4906	0.4909	0.4911	0.4913	0.4916
2.4	0.4918	0.4920	0.4922	0.4925	0.4927	0.4929	0.4931	0.4932	0.4934	0.4936
2.5	0.4938	0.4940	0.4941	0.4943	0.4945	0.4946	0.4948	0.4949	0.4951	0.4952
2.6	0.4953	0.4955	0.4956	0.4957	0.4959	0.4960	0.4961	0.4962	0.4963	0.4964
2.7	0.4965	0.4966	0.4967	0.4968	0.4969	0.4970	0.4971	0.4972	0.4973	0.4974
2.8	0.4974	0.4975	0.4976	0.4977	0.4977	0.4978	0.4979	0.4979	0.4980	0.4981
2.9	0.4981	0.4982	0.4982	0.4983	0.4984	0.4984	0.4985	0.4985	0.4986	0.4986
3.0	0.4987	0.4987	0.4987	0.4988	0.4988	0.4989	0.4989	0.4989	0.4990	0.4990

Source: R.J. Tallarida and R.B. Murray, *Manual of Pharmacologic Calculations with Computer Programs,* 2nd ed., New York: Springer-Verlag, 1987. With permission.

© 2003 by CRC Press LLC

TABLE 2 Poisson Distribution

Each number in this table represents the probability of obtaining at least X successes, or the area under the histogram to the right of and including the rectangle whose center is at X.

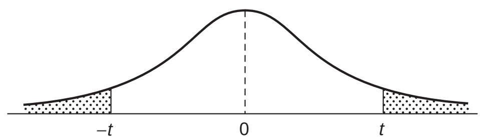


m	$X = 0$	$X = 1$	$X = 2$	$X = 3$	$X = 4$	$X = 5$	$X = 6$	$X = 7$	$X = 8$	$X = 9$	$X = 10$	$X = 11$	$X = 12$	$X = 13$	$X = 14$
.10	1.000	.095	.005												
.20	1.000	.181	.018	.001											
.30	1.000	.259	.037	.004											
.40	1.000	.330	.062	.008	.001										
.50	1.000	.393	.090	.014	.002										
.60	1.000	.451	.122	.023	.003										
.70	1.000	.503	.156	.034	.006	.001									
.80	1.000	.551	.191	.047	.009	.001									
.90	1.000	.593	.228	.063	.013	.002									
1.00	1.000	.632	.264	.080	.019	.004	.001								
1.1	1.000	.667	.301	.100	.026	.005	.001								
1.2	1.000	.699	.337	.120	.034	.008	.002								
1.3	1.000	.727	.373	.143	.043	.011	.002								
1.4	1.000	.753	.408	.167	.054	.014	.003	.001							
1.5	1.000	.777	.442	.191	.066	.019	.004	.001							
1.6	1.000	.798	.475	.217	.079	.024	.006	.001							
1.7	1.000	.817	.507	.243	.093	.030	.008	.002							
1.8	1.000	.835	.537	.269	.109	.036	.010	.003	.001						
1.9	1.000	.850	.566	.296	.125	.044	.013	.003	.001						
2.0	1.000	.865	.594	.323	.143	.053	.017	.005	.001						
2.2	1.000	.889	.645	.377	.181	.072	.025	.007	.002						
2.4	1.000	.909	.692	.430	.221	.096	.036	.012	.003	.001					
2.6	1.000	.926	.733	.482	.264	.123	.049	.017	.005	.001					
2.8	1.000	.939	.769	.531	.308	.152	.065	.024	.008	.002	.001				
3.0	1.000	.950	.801	.577	.353	.185	.084	.034	.012	.004	.001				
3.2	1.000	.959	.829	.620	.397	.219	.105	.045	.017	.006	.002				
3.4	1.000	.967	.853	.660	.442	.256	.129	.058	.023	.008	.003	.001			
3.6	1.000	.973	.874	.697	.485	.294	.156	.073	.031	.012	.004	.001			
3.8	1.000	.978	.893	.731	.527	.332	.184	.091	.040	.016	.006	.002			
4.0	1.000	.982	.908	.762	.567	.371	.215	.111	.051	.021	.008	.003	.001		
4.2	1.000	.985	.922	.790	.605	.410	.247	.133	.064	.028	.011	.004	.001		
4.4	1.000	.988	.934	.815	.641	.449	.280	.156	.079	.036	.015	.006	.002	.001	
4.6	1.000	.990	.944	.837	.674	.487	.314	.182	.095	.045	.020	.008	.003	.001	
4.8	1.000	.992	.952	.857	.706	.524	.349	.209	.113	.056	.025	.010	.004	.001	
5.0	1.000	.993	.960	.875	.735	.560	.384	.238	.133	.068	.032	.014	.005	.002	.001

Source: H.L. Adler and E.B. Roessler, *Introduction to Probability and Statistics,* 6th ed., New York: W. H. Freeman, 1977. With permission.

© 2003 by CRC Press LLC

TABLE 3 *t*-Distribution

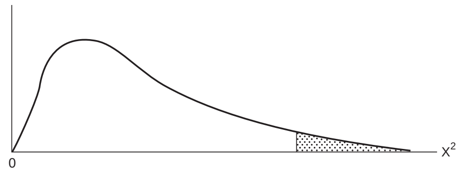


deg. freedom, f	90% ($P = 0.1$)	95% ($P = 0.05$)	99% ($P = 0.01$)
1	6.314	12.706	63.657
2	2.920	4.303	9.925
3	2.353	3.182	5.841
4	2.132	2.776	4.604
5	2.015	2.571	4.032
6	1.943	2.447	3.707
7	1.895	2.365	3.499
8	1.860	2.306	3.355
9	1.833	2.262	3.250
10	1.812	2.228	3.169
11	1.796	2.201	3.106
12	1.782	2.179	3.055
13	1.771	2.160	3.012
14	1.761	2.145	2.977
15	1.753	2.131	2.947
16	1.746	2.120	2.921
17	1.740	2.110	2.898
18	1.734	2.101	2.878
19	1.729	2.093	2.861
20	1.725	2.086	2.845
21	1.721	2.080	2.831
22	1.717	2.074	2.819
23	1.714	2.069	2.807
24	1.711	2.064	2.797
25	1.708	2.060	2.787
26	1.706	2.056	2.779
27	1.703	2.052	2.771
28	1.701	2.048	2.763
29	1.699	2.045	2.756
inf.	1.645	1.960	2.576

Source: R.J. Tallarida and R.B. Murray, *Manual of Pharmacologic Calculations with Computer Programs,* 2nd ed., New York: Springer-Verlag, 1987. With permission.

© 2003 by CRC Press LLC

TABLE 4 χ^2-Distribution



v	0.05	0.025	0.01	0.005
1	3.841	5.024	6.635	7.879
2	5.991	7.378	9.210	10.597
3	7.815	9.348	11.345	12.838
4	9.488	11.143	13.277	14.860
5	11.070	12.832	15.086	16.750
6	12.592	14.449	16.812	18.548
7	14.067	16.013	18.475	20.278
8	15.507	17.535	20.090	21.955
9	16.919	19.023	21.666	23.589
10	18.307	20.483	23.209	25.188
11	19.675	21.920	24.725	26.757
12	21.026	23.337	26.217	28.300
13	22.362	24.736	27.688	29.819
14	23.685	26.119	29.141	31.319
15	24.996	27.488	30.578	32.801
16	26.296	28.845	32.000	34.267
17	27.587	30.191	33.409	35.718
18	28.869	31.526	34.805	37.156
19	30.144	32.852	36.191	38.582
20	31.410	34.170	37.566	39.997
21	32.671	35.479	38.932	41.401
22	33.924	36.781	40.289	42.796
23	35.172	38.076	41.638	44.181
24	36.415	39.364	42.980	45.558
25	37.652	40.646	44.314	46.928
26	38.885	41.923	45.642	48.290
27	40.113	43.194	46.963	49.645
28	41.337	44.461	48.278	50.993
29	42.557	45.722	49.588	52.336
30	43.773	46.979	50.892	53.672

Source: J.E. Freund and F.J. Williams, *Elementary Business Statistics: The Modern Approach,* 2nd ed., Englewood Cliffs, N.J.: Prentice-Hall, 1972. With permission.

© 2003 by CRC Press LLC

TABLE 5 Variance Ratio

n_2	n_1 = 1	2	3	4	5	6	8	12	24	∞
					F(95%)					
1	161.4	199.5	215.7	224.6	230.2	234.0	238.9	243.9	249.0	254.3
2	18.51	19.00	19.16	19.25	19.30	19.33	19.37	19.41	19.45	19.50
3	10.13	9.55	9.28	9.12	9.01	8.94	8.84	8.74	8.64	8.53
4	7.71	6.94	6.59	6.39	6.26	6.16	6.04	5.91	5.77	5.63
5	6.61	5.79	5.41	5.19	5.05	4.95	4.82	4.68	4.53	4.36
6	5.99	5.14	4.76	4.53	4.39	4.28	4.15	4.00	3.84	3.67
7	5.59	4.74	4.35	4.12	3.97	3.87	3.73	3.57	3.41	3.23
8	5.32	4.46	4.07	3.84	3.69	3.58	3.44	3.28	3.12	2.93
9	5.12	4.26	3.86	3.63	3.48	3.37	3.23	3.07	2.90	2.71
10	4.96	4.10	3.71	3.48	3.33	3.22	3.07	2.91	2.74	2.54
11	4.84	3.98	3.59	3.36	3.20	3.09	2.95	2.79	2.61	2.40
12	4.75	3.88	3.49	3.26	3.11	3.00	2.85	2.69	2.50	2.30
13	4.67	3.80	3.41	3.18	3.02	2.92	2.77	2.60	2.42	2.21
14	4.60	3.74	3.34	3.11	2.96	2.85	2.70	2.53	2.35	2.13
15	4.54	3.68	3.29	3.06	2.90	2.79	2.64	2.48	2.29	2.07
16	4.49	3.63	3.24	3.01	2.85	2.74	2.59	2.42	2.24	2.01
17	4.45	3.59	3.20	2.96	2.81	2.70	2.55	2.38	2.19	1.96
18	4.41	3.55	3.16	2.93	2.77	2.66	2.51	2.34	2.15	1.92
19	4.38	3.52	3.13	2.90	2.74	2.63	2.48	2.31	2.11	1.88
20	4.35	3.49	3.10	2.87	2.71	2.60	2.45	2.28	2.08	1.84
21	4.32	3.47	3.07	2.84	2.68	2.57	2.42	2.25	2.05	1.81
22	4.30	3.44	3.05	2.82	2.66	2.55	2.40	2.23	2.03	1.78
23	4.28	3.42	3.03	2.80	2.64	2.53	2.38	2.20	2.00	1.76
24	4.26	3.40	3.01	2.78	2.62	2.51	2.36	2.18	1.98	1.73
25	4.24	3.38	2.99	2.76	2.60	2.49	2.34	2.16	1.96	1.71
26	4.22	3.37	2.98	2.74	2.59	2.47	2.32	2.15	1.95	1.69
27	4.21	3.35	2.96	2.73	2.57	2.46	2.30	2.13	1.93	1.67
28	4.20	3.34	2.95	2.71	2.56	2.44	2.29	2.12	1.91	1.65
29	4.18	3.33	2.93	2.70	2.54	2.43	2.28	2.10	1.90	1.64
30	4.17	3.32	2.92	2.69	2.53	2.42	2.27	2.09	1.89	1.62
40	4.08	3.23	2.84	2.61	2.45	2.34	2.18	2.00	1.79	1.51
60	4.00	3.15	2.76	2.52	2.37	2.25	2.10	1.92	1.70	1.39
120	3.92	3.07	2.68	2.45	2.29	2.17	2.02	1.83	1.61	1.25
∞	3.84	2.99	2.60	2.37	2.21	2.10	1.94	1.75	1.52	1.00
					F(99%)					
1	4052	4999	5403	5625	5764	5859	5982	6106	6234	6366
2	98.50	99.00	99.17	99.25	99.30	99.33	99.37	99.42	99.46	99.50
3	34.12	30.82	29.46	28.71	28.24	27.91	27.49	27.05	26.60	26.12
4	21.20	18.00	16.69	15.98	15.52	15.21	14.80	14.37	13.93	13.46
5	16.26	13.27	12.06	11.39	10.97	10.67	10.29	9.89	9.47	9.02
6	13.74	10.92	9.78	9.15	8.75	8.47	8.10	7.72	7.31	6.88
7	12.25	9.55	8.45	7.85	7.46	7.19	6.84	6.47	6.07	5.65
8	11.26	8.65	7.59	7.01	6.63	6.37	6.03	5.67	5.28	4.86
9	10.56	8.02	6.99	6.42	6.06	5.80	5.47	5.11	4.73	4.31
10	10.04	7.56	6.55	5.99	5.64	5.39	5.06	4.71	4.33	3.91
11	9.65	7.20	6.22	5.67	5.32	5.07	4.74	4.40	4.02	3.60
12	9.33	6.93	5.95	5.41	5.06	4.82	4.50	4.16	3.78	3.36
13	9.07	6.70	5.74	5.20	4.86	4.62	4.30	3.96	3.59	3.16
14	8.86	6.51	5.56	5.03	4.69	4.46	4.14	3.80	3.43	3.00
15	8.68	6.36	5.42	4.89	4.56	4.32	4.00	3.67	3.29	2.87
16	8.53	6.23	5.29	4.77	4.44	4.20	3.89	3.55	3.18	2.75
17	8.40	6.11	5.18	4.67	4.34	4.10	3.79	3.45	3.08	2.65
18	8.28	6.01	5.09	4.58	4.25	4.01	3.71	3.37	3.00	2.57

© 2003 by CRC Press LLC

TABLE 5 (continued) Variance Ratio

n_2	n_1: 1	2	3	4	5	6	8	12	24	∞
19	8.18	5.93	5.01	4.50	4.17	3.94	3.63	3.30	2.92	2.49
20	8.10	5.85	4.94	4.43	4.10	3.87	3.56	3.23	2.86	2.42
21	8.02	5.78	4.87	4.37	4.04	3.81	3.51	3.17	2.80	2.36
22	7.94	5.72	4.82	4.31	3.99	3.76	3.45	3.12	2.75	2.31
23	7.88	5.66	4.76	4.26	3.94	3.71	3.41	3.07	2.70	2.26
24	7.82	5.61	4.72	4.22	3.90	3.67	3.36	3.03	2.66	2.21
25	7.77	5.57	4.68	4.18	3.86	3.63	3.32	2.99	2.62	2.17
26	7.72	5.53	4.64	4.14	3.82	3.59	3.29	2.96	2.58	2.13
27	7.68	5.49	4.60	4.11	3.78	3.56	3.26	2.93	2.55	2.10
28	7.64	5.45	4.57	4.07	3.75	3.53	3.23	2.90	2.52	2.06
29	7.60	5.42	4.54	4.04	3.73	3.50	3.20	2.87	2.49	2.03
30	7.56	5.39	4.51	4.02	3.70	3.47	3.17	2.84	2.47	2.01
40	7.31	5.18	4.31	3.83	3.51	3.29	2.99	2.66	2.29	1.80
60	7.08	4.98	4.13	3.65	3.34	3.12	2.82	2.50	2.12	1.60
120	6.85	4.79	3.95	3.48	3.17	2.96	2.66	2.34	1.95	1.38
∞	6.64	4.60	3.78	3.32	3.02	2.80	2.51	2.18	1.79	1.00

Source: R.A. Fisher and F. Yates, *Statistical Tables for Biological, Agricultural and Medical Research,* London: The Lingman Group, Ltd. With permission.

Table of Derivatives

In the following table, *a* and *n* are constants, *e* is the base of the natural logarithms, and *u* and *v* denote functions of *x*.

1. $\frac{d}{dx}(a) = 0$

2. $\frac{d}{dx}(x) = 1$

3. $\frac{d}{dx}(au) = a\frac{du}{dx}$

4. $\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}$

5. $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

6. $\frac{d}{dx}(u/v) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

7. $\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}$

8. $\frac{d}{dx}e^u = e^u\frac{du}{dx}$

© 2003 by CRC Press LLC

9. $\dfrac{d}{dx}a^u = (\log_e a)a^u \dfrac{du}{dx}$

10. $\dfrac{d}{dx}\log_e u = (1/u)\dfrac{du}{dx}$

11. $\dfrac{d}{dx}\log_a u = (\log_a e)(1/u)\dfrac{du}{dx}$

12. $\dfrac{d}{dx}u^v = vu^{v-1}\dfrac{du}{dx} + u^v(\log_e u)\dfrac{dv}{dx}$

13. $\dfrac{d}{dx}\sin u = \cos u \dfrac{du}{dx}$

14. $\dfrac{d}{dx}\cos u = -\sin u \dfrac{du}{dx}$

15. $\dfrac{d}{dx}\tan u = \sec^2 u \dfrac{du}{dx}$

16. $\dfrac{d}{dx}\mathrm{ctn}\, u = -\csc^2 u \dfrac{du}{dx}$

17. $\dfrac{d}{dx}\sec u = \sec u \tan u \dfrac{du}{dx}$

18. $\dfrac{d}{dx}\csc u = -\mathrm{csu}\ \mathrm{ctn}\ u \dfrac{du}{dx}$

19. $\dfrac{d}{dx}\sin^{-1} u = \dfrac{1}{\sqrt{1-u^2}}\dfrac{du}{dx} \qquad (-\tfrac{1}{2}\pi \le \sin^{-1} u \le \tfrac{1}{2}\pi)$

20. $\dfrac{d}{dx}\cos^{-1} u = \dfrac{-1}{\sqrt{1-u^2}}\dfrac{du}{dx} \qquad (0 \le \cos^{-1} u \le \pi)$

21. $\dfrac{d}{dx}\tan^{-1} u = \dfrac{1}{1+u^2}\dfrac{du}{dx}$

22. $\dfrac{d}{dx}\mathrm{ctn}^{-1} u = \dfrac{-1}{1+u^2}\dfrac{du}{dx}$

23. $\dfrac{d}{dx}\sec^{-1} u = \dfrac{1}{u\sqrt{u^2-1}}\dfrac{du}{dx}$

$(-\pi \le \sec^{-1} u < -\tfrac{1}{2}\pi;\ 0 \le \sec^{-1} u < \tfrac{1}{2}\pi)$

© 2003 by CRC Press LLC

24. $\dfrac{d}{dx}\csc^{-1} u = \dfrac{-1}{u\sqrt{u^2-1}}\dfrac{du}{dx}$

$(-\pi < \csc^{-1} u \le -\frac{1}{2}\pi;\ 0 < \csc^{-1} u \le \frac{1}{2}\pi)$

25. $\dfrac{d}{dx}\sinh u = \cosh u \dfrac{du}{dx}$

26. $\dfrac{d}{dx}\cosh u = \sinh u \dfrac{du}{dx}$

27. $\dfrac{d}{dx}\tanh u = \operatorname{sech}^2 u \dfrac{du}{dx}$

28. $\dfrac{d}{dx}\operatorname{ctnh} u = -\operatorname{csch}^2 u \dfrac{du}{dx}$

29. $\dfrac{d}{dx}\operatorname{sech} u = -\operatorname{sech} u \tanh u \dfrac{du}{dx}$

30. $\dfrac{d}{dx}\operatorname{csch} u = -\operatorname{csch} u \operatorname{ctnh} u \dfrac{du}{dx}$

31. $\dfrac{d}{dx}\sinh^{-1} u = \dfrac{1}{\sqrt{u^2+1}}\dfrac{du}{dx}$

32. $\dfrac{d}{dx}\cosh^{-1} u = \dfrac{1}{\sqrt{u^2-1}}\dfrac{du}{dx}$

33. $\dfrac{d}{dx}\tanh^{-1} u = \dfrac{1}{1-u^2}\dfrac{du}{dx}$

34. $\dfrac{d}{dx}\operatorname{ctnh}^{-1} u = \dfrac{-1}{u^2-1}\dfrac{du}{dx}$

35. $\dfrac{d}{dx}\operatorname{sech}^{-1} u = \dfrac{-1}{u\sqrt{1-u^2}}\dfrac{du}{dx}$

36. $\dfrac{d}{dx}\operatorname{csch}^{-1} u = \dfrac{-1}{u\sqrt{u^2+1}}\dfrac{du}{dx}$

Additional Relations with Derivatives

$$\frac{d}{dt}\int_a^t f(x)\,dx = f(t) \qquad \frac{d}{dt}\int_t^a f(x)\,dx = -f(t)$$

If $x = f(y)$, then $\dfrac{dy}{dx} = \dfrac{1}{\dfrac{dx}{dy}}$

© 2003 by CRC Press LLC

If $y = f(u)$ and $u = g(x)$, then $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx}$ (chain rule)

If $x = f(t)$ and $y = g(t)$, then $\dfrac{dy}{dx} = \dfrac{g'(t)}{f'(t)}$, and $\dfrac{d^2y}{dx^2} = \dfrac{f'(t)g''(t) - g'(t)f''(t)}{[f'(t)]^3}$

(*Note*: Exponent in denominator is 3.)

Integrals

Elementary Forms

1. $\int a\,dx = ax$

2. $\int a \cdot f(x)\,dx = a\int f(x)\,dx$

3. $\int \phi(y)\,dx = \int \dfrac{\phi(y)}{y'}\,dy$, where $y' = \dfrac{dy}{dx}$

4. $\int (u+v)\,dx = \int u\,dx + \int v\,dx$, where u and v are any functions of x

5. $\int u\,dv = u\int dv - \int v\,du = uv - \int v\,du$

6. $\int u \dfrac{dv}{dx}\,dx = uv - \int v \dfrac{du}{dx}\,dx$

7. $\int x^n\,dx = \dfrac{x^{n+1}}{n+1}$, except $n = -1$

8. $\int \dfrac{f'(x)\,dx}{f(x)} = \log f(x)$ $(df(x) = f'(x)dx)$

9. $\int \dfrac{dx}{x} = \log x$

10. $\int \dfrac{f'(x)\,dx}{2\sqrt{f(x)}} = \sqrt{f(x)}$ $(df(x) = f'(x)dx)$

11. $\int e^x\,dx = e^x$

12. $\int e^{ax}\,dx = e^{ax}/a$

13. $\int b^{ax}\,dx = \dfrac{b^{ax}}{a\log b}$ $(b > 0)$

14. $\int \log x\,dx = x\log x - x$

© 2003 by CRC Press LLC

15. $\int a^x \log a \, dx = a^x \qquad (a > 0)$

16. $\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a}$

17. $$\int \frac{dx}{a^2 - x^2} = \begin{cases} \frac{1}{a} \tanh^{-1} \frac{x}{a} \\ \quad \text{or} \\ \frac{1}{2a} \log \frac{a + x}{a - x} \qquad (a^2 > x^2) \end{cases}$$

18. $$\int \frac{dx}{x^2 - a^2} = \begin{cases} -\frac{1}{a} \coth^{-1} \frac{x}{a} \\ \quad \text{or} \\ \frac{1}{2a} \log \frac{x - a}{x + a} \qquad (x^2 > a^2) \end{cases}$$

19. $$\int \frac{dx}{\sqrt{a^2 - x^2}} = \begin{cases} \sin^{-1} \frac{x}{|a|} \\ \quad \text{or} \\ -\cos^{-1} \frac{x}{|a|} \qquad (a^2 > x^2) \end{cases}$$

20. $\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \log\left(x + \sqrt{x^2 \pm a^2} \right)$

21. $\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{|a|} \sec^{-1} \frac{x}{a}$

22. $\int \frac{dx}{x\sqrt{a^2 \pm x^2}} = -\frac{1}{a} \log\left(\frac{a + \sqrt{a^2 \pm x^2}}{x} \right)$

Forms Containing $(a + bx)$

For forms containing $a + bx$ but not listed in the table, the substitution $u = \frac{a + bx}{x}$ may prove helpful.

23. $\int (a + bx)^n dx = \frac{(a + bx)^{n+1}}{(n + 1)b} \qquad (n \neq -1)$

24. $\int x(a + bx)^n dx = \frac{1}{b^2(n + 2)}(a + bx)^{n+2} - \frac{a}{b^2(n + 1)}(a + bx)^{n+1} \qquad (n \neq -1, -2)$

25. $\int x^2(a + bx)^n dx = \frac{1}{b^3}\left[\frac{(a + bx)^{n+3}}{n + 3} - 2a\frac{(a + bx)^{n+2}}{n + 2} + a^2\frac{(a + bx)^{n+1}}{n + 1} \right]$

© 2003 by CRC Press LLC

$$\text{26. } \int x^m(a+bx)^n dx = \begin{cases} \dfrac{x^{m+1}(a+bx)^n}{m+n+1} + \dfrac{an}{m+n+1}\displaystyle\int x^m(a+bx)^{n-1}dx \\ \qquad\text{or} \\ \dfrac{1}{a(n+1)}\left[-x^{m+1}(a+bx)^{n+1} + (m+n+2)\displaystyle\int x^m(a+bx)^{n+1}dx\right] \\ \qquad\text{or} \\ \dfrac{1}{b(m+n+1)}\left[x^m(a+bx)^{n+1} - ma\displaystyle\int x^{m-1}(a+bx)^n dx\right] \end{cases}$$

$$\text{27. } \int \frac{dx}{a+bx} = \frac{1}{b}\log(a+bx)$$

$$\text{28. } \int \frac{dx}{(a+bx)^2} = -\frac{1}{b(a+bx)}$$

$$\text{29. } \int \frac{dx}{(a+bx)^3} = -\frac{1}{2b(a+bx)^2}$$

$$\text{30. } \int \frac{x\,dx}{a+bx} = \begin{cases} \dfrac{1}{b^2}[a+bx-a\log(a+bx)] \\ \qquad\text{or} \\ \dfrac{x}{b} - \dfrac{a}{b^2}\log(a+bx) \end{cases}$$

$$\text{31. } \int \frac{x\,dx}{(a+bx)^2} = \frac{1}{b^2}\left[\log(a+bx) + \frac{a}{a+bx}\right]$$

$$\text{32. } \int \frac{x\,dx}{(a+bx)^n} = \frac{1}{b^2}\left[\frac{-1}{(n-2)(a+bx)^{n-2}} + \frac{a}{(n-1)(a+bx)^{n-1}}\right] \qquad (n \neq 1, 2)$$

$$\text{33. } \int \frac{x^2 dx}{a+bx} = \frac{1}{b^3}\left[\frac{1}{2}(a+bx)^2 - 2a(a+bx) + a^2\log(a+bx)\right]$$

$$\text{34. } \int \frac{x^2 dx}{(a+bx)^2} = \frac{1}{b^3}\left[a+bx-2a\log(a+bx) - \frac{a^2}{a+bx}\right]$$

$$\text{35. } \int \frac{x^2 dx}{(a+bx)^3} = \frac{1}{b^3}\left[\log(a+bx) + \frac{2a}{a+bx} - \frac{a^2}{2(a+bx)^2}\right]$$

$$\text{36. } \int \frac{x^2 dx}{(a+bx)^n} = \frac{1}{b^3}\left[\frac{-1}{(n-3)(a+bx)^{n-3}} + \frac{2a}{(n-2)(a+bx)^{n-2}} - \frac{a^2}{(n-1)(a+bx)^{n-1}}\right] \qquad (n \neq 1, 2, 3)$$

© 2003 by CRC Press LLC

37. $\int \frac{dx}{x(a+bx)} = -\frac{1}{a}\log\frac{a+bx}{x}$

38. $\int \frac{dx}{x(a+bx)^2} = \frac{1}{a(a+bx)} - \frac{1}{a^2}\log\frac{a+bx}{x}$

39. $\int \frac{dx}{x(a+bx)^3} = \frac{1}{a^3}\left[\frac{1}{2}\left(\frac{2a+bx}{a+bx}\right)^2 + \log\frac{x}{a+bx}\right]$

40. $\int \frac{dx}{x^2(a+bx)} = -\frac{1}{ax} + \frac{b}{a^2}\log\frac{a+bx}{x}$

41. $\int \frac{dx}{x^3(a+bx)} = \frac{2bx-a}{2a^2x^2} + \frac{b^2}{a^3}\log\frac{x}{a+bx}$

42. $\int \frac{dx}{x^2(a+bx)^2} = -\frac{a+2bx}{a^2x(a+bx)} + \frac{2b}{a^3}\log\frac{a+bx}{x}$

The Fourier Transforms

For a piecewise continuous function $F(x)$ over a finite interval $0 \leqq x \leqq \pi$, the *finite Fourier cosine transform* of $F(x)$ is

$$f_c(n) = \int_0^\pi F(x)\cos nx\,dx \qquad (n = 0, 1, 2, \ldots) \tag{1}$$

If x ranges over the interval $0 \leqq x \leqq L$, the substitution $x' = \pi x/L$ allows the use of this definition, also. The inverse transform is written

$$\bar{F}(x) = \frac{1}{\pi}f_c(0) + \frac{2}{\pi}\sum_{n=1}^{\infty} f_c(n)\cos nx \qquad (0 < x < \pi) \tag{2}$$

where $\bar{F}(x) = \frac{[F(x+0)+F(x-0)]}{2}$. We observe that $\bar{F}(x) = F(x)$ at points of continuity. The formula

$$\begin{aligned} f_c^{(2)}(n) &= \int_0^\pi F''(x)\cos nx\,dx \\ &= -n^2 f_c(n) - F'(0) + (-1)^n F'(\pi) \end{aligned} \tag{3}$$

makes the finite Fourier cosine transform useful in certain boundary value problems.

Analogously, the *finite Fourier sine transform* of $F(x)$ is

$$f_s(n) = \int_0^\pi F(x)\sin nx\,dx \qquad (n = 1, 2, 3, \ldots) \tag{4}$$

and

$$\bar{F}(x) = \frac{2}{\pi}\sum_{n=1}^{\infty} f_s(n)\sin nx \qquad (0 < x < \pi) \tag{5}$$

© 2003 by CRC Press LLC

Corresponding to (3) we have

$$\begin{aligned} f_s^{(2)}(n) &= \int_0^{\pi} F''(x)\sin nx\,dx \\ &= -n^2 f_s(n) - nF(0) - n(-1)^n F(\pi) \end{aligned} \tag{6}$$

Fourier Transforms

If $F(x)$ is defined for $x \geqq 0$ and is piecewise continuous over any finite interval, and if

$$\int_0^{\infty} F(x)dx$$

is absolutely convergent, then

$$f_c(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} F(x)\cos(\alpha x)dx \tag{7}$$

is the *Fourier cosine transform* of $F(x)$. Furthermore,

$$\bar{F}(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f_c(\alpha)\cos(\alpha x)d\alpha \tag{8}$$

If $\lim\limits_{x\to\infty} \frac{d^nF}{dx^n} = 0$, an important property of the Fourier cosine transform,

$$\begin{aligned} f_c^{(2r)}(\alpha) &= \sqrt{\frac{2}{\pi}} \int_0^{\infty} \left(\frac{d^{2r}F}{dx^{2r}}\right)\cos(\alpha x)dx \\ &= -\sqrt{\frac{2}{\pi}} \sum_{n=0}^{r-1} (-1)^n a_{2r-2n-1} a^{2n} + (-1)'\alpha^{2r} f_c(\alpha) \end{aligned} \tag{9}$$

where $\lim\limits_{x\to 0} \frac{d^rF}{dx^r} = a_r$, makes it useful in the solution of many problems.

Under the same conditions,

$$f_s(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} F(x)\sin(\alpha x)dx \tag{10}$$

defines the *Fourier sine transform* of $F(x)$, and

$$\bar{F}(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f_s(\alpha)\sin(\alpha x)d\alpha \tag{11}$$

Corresponding to (9), we have

$$\begin{aligned} f_s^{(2r)}(\alpha) &= \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{d^{2r}F}{dx^{2r}}\sin(ax)dx \\ &= -\sqrt{\frac{2}{\pi}} \sum_{n=1}^{r} (-1)^n \alpha^{2n-1} a_{2r-2n} + (-1)^{r-1}\alpha^{2r} f_s(\alpha) \end{aligned} \tag{12}$$

Similarly, if $F(x)$ is defined for $-\infty < x < \infty$, and if $\int_{-\infty}^{\infty} F(x)\,dx$ is absolutely convergent, then

$$f(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(x)e^{i\alpha x}dx \tag{13}$$

© 2003 by CRC Press LLC

is the *Fourier transform* of $F(x)$, and

$$\bar{F}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\alpha) e^{-i\alpha x} d\alpha \tag{14}$$

Also, if

$$\lim_{|x| \to \infty} \left| \frac{d^n F}{dx^n} \right| = 0 \qquad (n = 1, 2, \ldots, r-1)$$

then

$$f^{(r)}(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F^{(r)}(x) e^{i\alpha x} dx = (-i\alpha)^r f(\alpha) \tag{15}$$

Finite Sine Transforms

	$f_s(n)$	$F(x)$
1	$f_s(n) = \int_0^{\pi} F(x) \sin nx\, dx \quad (n = 1, 2, \ldots)$	$F(x)$
2	$(-1)^{n+1} f_s(n)$	$F(\pi - x)$
3	$\frac{1}{n}$	$\frac{\pi - x}{\pi}$
4	$\frac{(-1)^{n+1}}{n}$	$\frac{x}{\pi}$
5	$\frac{1-(-1)^n}{n}$	1
6	$\frac{2}{n^2} \sin \frac{n\pi}{2}$	$\begin{cases} x & \text{when } 0 < x < \pi/2 \\ \pi - x & \text{when } \pi/2 < x < \pi \end{cases}$
7	$\frac{(-1)^{n+1}}{n^3}$	$\frac{x(\pi^2 - x^2)}{6\pi}$
8	$\frac{1-(-1)^n}{n^3}$	$\frac{x(\pi - x)}{2}$
9	$\frac{\pi^2(-1)^{n-1}}{n} - \frac{2[1-(-1)^n]}{n^3}$	x^2
10	$\pi(-1)^n \left(\frac{6}{n^3} - \frac{\pi^2}{n}\right)$	x^3
11	$\frac{n}{n^2 + c^2}[1 - (-1)^n e^{c\pi}]$	e^{cx}
12	$\frac{n}{n^2 + c^2}$	$\frac{\sinh c(\pi - x)}{\sinh c\pi}$
13	$\frac{n}{n^2 - k^2} \; (k \neq 0, 1, 2, \ldots)$	$\frac{\sin k(\pi - x)}{\sin k\pi}$
14	$\begin{cases} \frac{\pi}{2} \text{ when } n = m \\ 0 \text{ when } n \neq m \end{cases} \quad (m = 1, 2, \ldots)$	$\sin mx$

© 2003 by CRC Press LLC

	$f_s(n)$	$F(x)$
15	$\frac{n}{n^2-k^2}[1-(-1)^n \cos k\pi] \quad (k \neq 1, 2, \ldots)$	$\cos kx$
16	$\begin{cases} \frac{n}{n^2-m^2}[1-(-1)^{n+m}] \\ \quad \text{when } n \neq m = 1, 2, \ldots \\ 0 \quad \text{when } n = m \end{cases}$	$\cos mx$
17	$\frac{n}{(n^2-k^2)^2}(k \neq 0, 1, 2, \ldots)$	$\frac{\pi \sin kx}{2k \sin^2 k\pi} - \frac{x \cos k(\pi - x)}{2k \sin k\pi}$
18	$\frac{b^n}{n}(\lvert b\rvert \leq 1)$	$\frac{2}{\pi} \arctan \frac{b \sin x}{1 - b \cos x}$
19	$\frac{1-(-1)^n}{n} b^n \quad (\lvert b\rvert \leq 1)$	$\frac{2}{\pi} \arctan \frac{2b \sin x}{1 - b^2}$

Finite Cosine Transforms

	$f_c(n)$	$F(x)$
1	$f_c(n) = \int_0^\pi F(x) \cos nx \, dx \quad (n = 0, 1, 2, \ldots)$	$F(x)$
2	$(-1)^n f_c(n)$	$F(\pi - x)$
3	0 when $n = 1, 2, \ldots$; $f_c(0) = \pi$	1
4	$\frac{2}{n} \sin \frac{n\pi}{2}$; $f_c(0) = 0$	$\begin{cases} 1 \text{ when } 0 < x < \pi/2 \\ -1 \text{ when } \pi/2 < x < \pi \end{cases}$
5	$-\frac{1-(-1)^n}{n^2}$; $f_c(0) = \frac{\pi^2}{2}$	x
6	$\frac{(-1)^n}{n^2}$; $f_c(0) = \frac{\pi^2}{6}$	$\frac{x^2}{2\pi}$
7	$\frac{1}{n^2}$; $f_c(0) = 0$	$\frac{(\pi - x)^2}{2\pi} - \frac{\pi}{6}$
8	$3\pi^2 \frac{(-1)^n}{n^2} - 6\frac{1-(-1)^n}{n^4}$; $f_c(0) = \frac{\pi^4}{4}$	x^3
9	$\frac{(-1)^n e^c \pi - 1}{n^2 + c^2}$	$\frac{1}{c} e^{cx}$
10	$\frac{1}{n^2 + c^2}$	$\frac{\cosh c(\pi - x)}{c \sinh c\pi}$
11	$\frac{k}{n^2 - k^2}[(-1)^n \cos \pi k - 1] \quad (k \neq 0, 1, 2, \ldots)$	$\sin kx$
12	$\frac{(-1)^{n+m} - 1}{n^2 - m^2}$; $f_c(m) = 0 \quad (m = 1, 2, \ldots)$	$\frac{1}{m} \sin mx$
13	$\frac{1}{n^2 - k^2} \quad (k \neq 0, 1, 2, \ldots)$	$-\frac{\cos k(\pi - x)}{k \sin k\pi}$
14	0 when $n = 1, 2, \ldots$; $f_c(m) = \frac{\pi}{2} \quad (m = 1, 2, \ldots)$	$\cos mx$

© 2003 by CRC Press LLC

Fourier Sine Transforms

	$F(x)$	$f_s(\alpha)$
1	$\begin{cases} 1 & (0 < x < a) \\ 0 & (x > a) \end{cases}$	$\sqrt{\frac{2}{\pi}}\left[\frac{1-\cos\alpha}{\alpha}\right]$
2	$x^{p-1}(0 < p < 1)$	$\sqrt{\frac{2}{\pi}}\,\frac{\Gamma(p)}{\alpha^p}\sin\frac{p\pi}{2}$
3	$\begin{cases} \sin x & (0 < x < a) \\ 0 & (x > a) \end{cases}$	$\frac{1}{\sqrt{2\pi}}\left[\frac{\sin[a(1-\alpha)]}{1-\alpha} - \frac{\sin[a(1+\alpha)]}{1+\alpha}\right]$
4	e^{-x}	$\sqrt{\frac{2}{\pi}}\left[\frac{\alpha}{1+\alpha^2}\right]$
5	$xe^{-x^2/2}$	$\alpha e^{-\alpha^2/2}$
6	$\cos\frac{x^2}{2}$	$\sqrt{2}\left[\sin\frac{\alpha^2}{2}C\left(\frac{\alpha^2}{2}\right) - \cos\frac{\alpha^2}{2}S\left(\frac{\alpha^2}{2}\right)\right]^*$
7	$\sin\frac{x^2}{2}$	$\sqrt{2}\left[\cos\frac{\alpha^2}{2}C\left(\frac{\alpha^2}{2}\right) + \sin\frac{\alpha^2}{2}S\left(\frac{\alpha^2}{2}\right)\right]^*$

* $C(y)$ and $S(y)$ are the Fresnel integrals.

$$C(y) = \frac{1}{\sqrt{2\pi}}\int_0^y \frac{1}{\sqrt{t}}\cos t\,dt$$

$$S(y) = \frac{1}{\sqrt{2\pi}}\int_0^y \frac{1}{\sqrt{t}}\sin t\,dt$$

Fourier Cosine Transforms

	$F(x)$	$f_c(\alpha)$
1	$\begin{cases} 1 & (0 < x < a) \\ 0 & (x > a) \end{cases}$	$\sqrt{\frac{2}{\pi}}\,\frac{\sin a\alpha}{\alpha}$
2	$x^{p-1}\ (0 < p < 1)$	$\sqrt{\frac{2}{\pi}}\,\frac{\Gamma(p)}{\alpha^p}\cos\frac{p\pi}{2}$
3	$\begin{cases} \cos x & (0 < x < a) \\ 0 & (x > a) \end{cases}$	$\frac{1}{\sqrt{2\pi}}\left[\frac{\sin[a(1-\alpha)]}{1-\alpha} + \frac{\sin[a(1+\alpha)]}{1+\alpha}\right]$
4	e^{-x}	$\sqrt{\frac{2}{\pi}}\left(\frac{1}{1+\alpha^2}\right)$
5	$e^{-x^2/2}$	$e^{-\alpha^2/2}$
6	$\cos\frac{x^2}{2}$	$\cos\left(\frac{\alpha^2}{2} - \frac{\pi}{4}\right)$
7	$\sin\frac{x^2}{2}$	$\cos\left(\frac{\alpha^2}{2} + \frac{\pi}{4}\right)$

© 2003 by CRC Press LLC

Fourier Transforms

	$F(x)$	$f(\alpha)$
1	$\dfrac{\sin ax}{x}$	$\begin{cases} \sqrt{\dfrac{\pi}{2}} & \lvert\alpha\rvert < a \\ 0 & \lvert\alpha\rvert > a \end{cases}$
2	$\begin{cases} e^{iwx} & (p < x < q) \\ 0 & (x < p,\ x > q) \end{cases}$	$\dfrac{i}{\sqrt{2\pi}} \dfrac{e^{ip(w+\alpha)} - e^{iq(w+\alpha)}}{(w+\alpha)}$
3	$\begin{cases} e^{-cx+iwx} & (x > 0) \\ 0 & (x < 0) \end{cases} \quad (c > 0)$	$\dfrac{i}{\sqrt{2\pi}(w+\alpha+ic)}$
4	$e^{-px^2} \quad R(p) > 0$	$\dfrac{1}{\sqrt{2p}} e^{-\alpha^2/4p}$
5	$\cos px^2$	$\dfrac{1}{\sqrt{2p}} \cos\left[\dfrac{\alpha^2}{4p} - \dfrac{\pi}{4}\right]$
6	$\sin px^2$	$\dfrac{1}{\sqrt{2p}} \cos\left[\dfrac{\alpha^2}{4p} + \dfrac{\pi}{4}\right]$
7	$\lvert x\rvert^{-p} \quad (0 < p < 1)$	$\sqrt{\dfrac{2}{\pi}} \dfrac{\Gamma(1-p)\sin\dfrac{p\pi}{2}}{\lvert\alpha\rvert^{(1-p)}}$
8	$\dfrac{e^{-a\lvert x\rvert}}{\sqrt{\lvert x\rvert}}$	$\dfrac{\sqrt{\sqrt{(a^2+\alpha^2)}+a}}{\sqrt{a^2+\alpha^2}}$
9	$\dfrac{\cosh ax}{\cosh \pi x} \quad (-\pi < a < \pi)$	$\sqrt{\dfrac{2}{\pi}} \dfrac{\cos\dfrac{a}{2}\cosh\dfrac{\alpha}{2}}{\cosh\alpha + \cos a}$
10	$\dfrac{\sinh ax}{\sinh \pi x} \quad (-\pi < a < \pi)$	$\dfrac{1}{\sqrt{2\pi}} \dfrac{\sin a}{\cosh\alpha + \cos a}$
11	$\begin{cases} \dfrac{1}{\sqrt{a^2-x^2}} & (\lvert x\rvert < a) \\ 0 & (\lvert x\rvert > a) \end{cases}$	$\sqrt{\dfrac{\pi}{2}} J_0(a\alpha)$
12	$\dfrac{\sin\left[b\sqrt{a^2+x^2}\right]}{\sqrt{a^2+x^2}}$	$\begin{cases} 0 & (\lvert\alpha\rvert > b) \\ \sqrt{\dfrac{\pi}{2}} J_0(a\sqrt{b^2-\alpha^2}) & (\lvert\alpha\rvert < b) \end{cases}$
13	$\begin{cases} P_n(x) & (\lvert x\rvert < 1) \\ 0 & (\lvert x\rvert > 1) \end{cases}$	$\dfrac{i^n}{\sqrt{\alpha}} J_{n+\frac{1}{2}}(\alpha)$
14	$\begin{cases} \dfrac{\cos\left[b\sqrt{a^2-x^2}\right]}{\sqrt{a^2-x^2}} & (\lvert x\rvert < a) \\ 0 & (\lvert x\rvert > a) \end{cases}$	$\sqrt{\dfrac{\pi}{2}} J_0(a\sqrt{a^2+b^2})$
15	$\begin{cases} \dfrac{\cosh\left[b\sqrt{a^2-x^2}\right]}{\sqrt{a^2-x^2}} & (\lvert x\rvert < a) \\ 0 & (\lvert x\rvert > a) \end{cases}$	$\sqrt{\dfrac{\pi}{2}} J_0(a\sqrt{\alpha^2-b^2})$

© 2003 by CRC Press LLC

The following functions appear among the entries of the tables on transforms.

Function	Definition	Name
$Ei(x)$	$\int_{-\infty}^{x} \frac{e^v}{v} dv$; or sometimes defined as $-Ei(-x) = \int_{x}^{\infty} \frac{e^{-v}}{v} dv$	
$Si(x)$	$\int_{0}^{x} \frac{\sin v}{v} dv$	
$Ci(x)$	$\int_{\infty}^{x} \frac{\cos v}{v} dv$; or sometimes defined as negative of this integral	
$erf(x)$	$\frac{2}{\sqrt{\pi}} \int_{0}^{x} e^{-v^2} dv$	Error function
$erfc(x)$	$1 - erf(x) = \frac{2}{\sqrt{\pi}} \int_{x}^{\infty} e^{-v^2} dv$	Complementary function to error function
$L_n(x)$	$\frac{e^x}{n!} \frac{d^n}{dx^n}(x^n e^{-x}) \quad (n = 0, 1, 2, \ldots)$	Laguerre polynomial of degree n

Numerical Methods

Solution of Equations by Iteration

Fixed-Point Iteration for Solving $f(x) = 0$

Transform $f(x) = 0$ into the form $x = g(x)$. Choose x_0 and compute $x_1 = g(x_0)$, $x_2 = g(x_1)$, and in general

$$x_{n+1} = gx_n \qquad (n = 0, 1, 2, \ldots)$$

Newton–Raphson Method for Solving $f(x) = 0$

f is assumed to have a continuous derivative f'. Use an approximate value x_0 obtained from the graph of f. Then compute

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}, \qquad x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

and in general

$$x_{n+1} = x_n - \frac{f(x_0)}{f'(x_n)}$$

Secant Method for Solving $f(x) = 0$

The secant method is obtained from Newton's method by replacing the derivative $f'(x)$ by the difference quotient

$$f'(x_n) = \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}$$

Thus,

$$x_{n+1} = x_n - f(x_n)\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}$$

The secant method needs two starting values x_0 and x_1.

© 2003 by CRC Press LLC

Method of Regula Falsi for Solving $f(x) = 0$

Select two starting values x_0 and x_1. Then compute

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$$

If $f(x_0) \cdot f(x_2) < 0$, replace x_1 by x_2 in formula for x_2, leaving x_0 unchanged, and then compute the next approximation x_3; otherwise, replace x_0 by x_2, leaving x_1 unchanged, and compute the next approximation x_3. Continue in a similar manner.

Finite Differences

Uniform Interval h

If a function $f(x)$ is tabulated at a uniform interval h, that is, for arguments given by $x_n = x_0 + nh$, where n is an integer, then the function $f(x)$ may be denoted by f_n.

This can be generalized so that for all values of p, and in particular for $0 \leqq p \leqq 1$,

$$f(x_0 + ph) = f(x_p) = f_p$$

where the argument designated x_0 can be chosen quite arbitrarily.

The following table lists and defines the standard operators used in numerical analysis.

Symbol	Function	Definition
E	Displacement	$Ef_p = f_{p+1}$
Δ	Forward difference	$\Delta f_p = f_{p+1} - f_p$
∇	Backward difference	$\nabla f_p = f_p - f_{p-1}$
A	Divided difference	
δ	Central difference	$of_p = f_{p+\frac{1}{2}} - f_{p-\frac{1}{2}}$
μ	Average	$uf_p = \frac{1}{2}\left(f_{p+\frac{1}{2}} + f_{p-\frac{1}{2}}\right)$
Δ^{-1}	Backward sum	$\Delta^{-1} f_p = \Delta^{-1} f_{p-1} + f_{p-1}$
∇^{-1}	Forward sum	$\nabla^{-1} f_p = \nabla^{-1} f_{p-1} + f_p$
δ^{-1}	Central sum	$\delta^{-1} f_p = \delta^{-1} f_{p-1} + f_{p-\frac{1}{2}}$
D	Differentiation	$Df_p = \frac{d}{dx} f(x) = \frac{1}{h} \cdot \frac{d}{dp} f_p$
$I\ (= D^{-1})$	Integration	$If_p = \int^{x_p} f(x)\,dx = h\int^{p} f_p\,dp$
$J\ (= \Delta D^{-1})$	Definite integration	$Jf_p = h\int_p^{p+1} f_p\,dp$

I, Δ^{-1}, ∇^{-1}, and δ^{-1} all imply the existence of an arbitrary constant that is determined by the initial conditions of the problem.

Where no confusion can arise, the f can be omitted as, for example, in writing Δ_p for Δf_p.

Higher differences are formed by successive operations, e.g.,

© 2003 by CRC Press LLC

$$\begin{aligned}
\Delta^2 f_p &= \Delta_p^2 \\
&= \Delta \cdot \Delta_p \\
&= \Delta(f_{p+1} - f_p) \\
&= \Delta_{p+1} - \Delta_p \\
&= f_{p+2} - f_{p+1} - f_{p+1} + f_p \\
&= f_{p+2} - 2f_{p+1} + f_p
\end{aligned}$$

Note that $f_p \equiv \Delta_p^0 \equiv \nabla_p^0 \equiv \delta_p^0$.

The disposition of the differences and sums relative to the function values is as shown (the arguments are omitted in these cases in the interest of clarity).

Calculus of Finite Differences

Forward difference scheme

Δ_{-1}^{-2}		f_{-2}		Δ_{-2}^{2}	
	Δ_{-1}^{-1}		Δ_{-2}		Δ_{-3}^{3}
Δ_{0}^{-2}		f_{-1}		Δ_{-2}^{2}	
	Δ_{0}^{-1}		Δ_{-1}		Δ_{-2}^{3}
Δ_{1}^{-2}		f_{0}		Δ_{-1}^{2}	
	Δ_{1}^{-1}		Δ_{0}		Δ_{-1}^{3}
Δ_{2}^{-2}		f_{1}		Δ_{0}^{2}	
	Δ_{2}^{-1}		Δ_{1}		Δ_{0}^{3}
Δ_{3}^{-2}		f_{2}		Δ_{1}^{2}	

Backward difference scheme

∇_{-3}^{-2}		f_{-2}		∇_{-1}^{2}	
	∇_{-2}^{-1}		∇_{-1}		∇_{0}^{2}
∇_{-2}^{-2}		f_{-1}		∇_{0}^{2}	
	∇_{-1}^{-1}		∇_{0}		∇_{1}^{3}
∇_{-1}^{-2}		f_{0}		∇_{1}^{2}	
	∇_{0}^{-1}		∇_{1}		∇_{2}^{3}
∇_{0}^{-2}		f_{1}		∇_{2}^{2}	
	∇^{-1}		∇_{2}		∇_{3}^{3}
∇_{1}^{-2}		f_{2}		∇_{3}^{2}	

Central difference scheme

δ_{-2}^{-2}		f_{-2}		δ_{-2}^{2}		δ_{-2}^{4}
	$\delta_{-1\frac{1}{2}}^{-1}$		$\delta_{-1\frac{1}{2}}$		$\delta_{-1\frac{1}{2}}^{3}$	
δ_{-1}^{-2}		f_{-1}		δ_{-1}^{2}		δ_{-1}^{4}
	$\delta_{-\frac{1}{2}}^{-1}$		$\delta_{-\frac{1}{2}}$		$\delta_{-\frac{1}{2}}^{3}$	
δ_{0}^{-2}		f_{0}		δ_{0}^{2}		δ_{0}^{4}
	$\delta_{\frac{1}{2}}^{-1}$		$\delta_{\frac{1}{2}}$		$\delta_{\frac{1}{2}}^{3}$	
δ_{1}^{-2}		f_{1}		δ_{1}^{2}		δ_{1}^{4}
	$\delta_{-1\frac{1}{2}}^{-1}$		$\delta_{1\frac{1}{2}}$		$\delta_{1\frac{1}{2}}^{3}$	
δ_{2}^{-2}		f_{2}		δ_{2}^{2}		δ_{2}^{4}

In the forward difference scheme, the subscripts are seen to move forward into the difference table and no fractional subscripts occur. In the backward difference scheme, the subscripts lie on diagonals slanting backward into the table, while in the central difference scheme, the subscripts maintain their positions and the odd-order subscripts are fractional.

All three, however, are merely alternative ways of labeling the same numerical quantities, as any difference is the result of subtracting the number diagonally above it in the preceding column from that diagonally below it in the preceding column, or, alternatively, it is the sum of the number diagonally above it in the subsequent column with that immediately above it in its own column.

© 2003 by CRC Press LLC

In general, $\Delta^n_{p-\frac{1}{2}n} \equiv \delta^n_p \equiv \nabla^n_{p+\frac{1}{2}n}$.

If a polynomial of degree r is tabulated exactly, i.e., without any round-off errors, then the rth differences are constant.

The following table enables the simpler operators to be expressed in terms of the others:

	E	Δ	δ, μ	∇
E	—	$1+\Delta$	$1+\mu\delta+\frac{1}{2}\delta^2$	$(1-\nabla)^{-1}$
Δ	$E-1$	—	$\mu\delta+\frac{1}{2}\delta^2$	$\nabla(1-\nabla)^{-1}$
δ	$E^{\frac{1}{2}}-E^{-\frac{1}{2}}$	$\Delta(1+\Delta)^{-\frac{1}{2}}$	$2(\mu^2-1)^{\frac{1}{2}}$	$\nabla(1-\nabla)^{-\frac{1}{2}}$
∇	$-E^{-1}$	$\Delta(1+\Delta)^{-1}$	$\mu\delta-\frac{1}{2}\delta^2$	—
μ	$\frac{1}{2}\left(E^{\frac{1}{2}}+E^{-\frac{1}{2}}\right)$	$\frac{1}{2}(2+\Delta)(1+\Delta)^{-\frac{1}{2}}$	$(1+\frac{1}{4}\delta^2)^{\frac{1}{2}}$	$\frac{1}{2}(2-\nabla)(1-\nabla)^{-\frac{1}{2}}$

In addition to the above, there are other identities by means of which the above table can be extended, such as

$$E = e^{hD} = \Delta\nabla^{-1}$$

$$\mu = E^{-\frac{1}{2}}+\frac{1}{2}\delta = E^{\frac{1}{2}}-\frac{1}{2}\delta = \cosh(\tfrac{1}{2}hD)$$

$$\delta = E^{-\frac{1}{2}}\Delta = E^{\frac{1}{2}}\nabla = (\Delta\nabla)^{\frac{1}{2}} = 2\sinh(\tfrac{1}{2}hD)$$

Note the emergence of Taylor's series from

$$\begin{aligned} f_p &= E^p f_0 \\ &= e^{phD} f_0 \\ &= f_0 + phDf_0 + \frac{1}{2!}p^2h^2D^2f_0 + \cdots \end{aligned}$$

Interpolation

Finite difference interpolation entails taking a given set of points and fitting a function to them. This function is usually a polynomial. If the graph of $f(x)$ is approximated over one tabular interval by a chord of the form $y = a + bx$ chosen to pass through the two points

$$(x_0, f(x_0)), \qquad (x_0+h, f(x_0+h))$$

the formula for the interpolated value is found to be

$$\begin{aligned} f(x_0+ph) &= f(x_0) + p[f(x_0+h) - f(x_0)] \\ &= f(x_0) + p\Delta f_0 \end{aligned}$$

If the graph of $f(x)$ is approximated over two successive tabular intervals by a parabola of the form $y = a + bx + cx^2$ chosen to pass through the three points

© 2003 by CRC Press LLC

$$(x_0, f(x_0)), \qquad (x_0 + h, f(x_0 + h)), \qquad (x_0 + 2h, f(x_0 + 2h))$$

the formula for the interpolated value is found to be

$$\begin{aligned} f(x_0 + ph) &= f(x_0) + p[f(x_0 + h) - f(x_0)] \\ &\quad + \frac{p(p-1)}{2!}[f(x_0 + 2h) - 2f(x_0 + h) + f(x_0)] \\ &= f_0 + p\Delta f_0 + \frac{p(p-1)}{2!}\Delta^2 f_0 \end{aligned}$$

Using polynomial curves of higher order to approximate the graph of $f(x)$, a succession of interpolation formulas involving higher differences of the tabulated function can be derived. These formulas provide, in general, higher accuracy in the interpolated values.

Newton's Forward Formula

$$f_p = f_0 + p\Delta_0 + \frac{1}{2!}p(p-1)\Delta_0^2 + \frac{1}{3!}p(p-1)(p-2)\Delta_0^3 \cdots \qquad 0 \le p \le 1$$

Newton's Backward Formula

$$f_p = f_0 + p\nabla_0 + \frac{1}{2!}p(p+1)\nabla_0^2 + \frac{1}{3!}p(p+1)(p+2)\nabla_0^3 \cdots \qquad 0 \le p \le 1$$

Gauss' Forward Formula

$$f_p = f_0 + p\delta_{\frac{1}{2}} + G_2\delta_0^2 + G_3\delta_{\frac{1}{2}}^3 + G_4\delta_0^4 + G_5\delta_{\frac{1}{2}}^5 \cdots \qquad 0 \le p \le 1$$

Gauss' Backward Formula

$$f_p = f_0 + p\delta_{-\frac{1}{2}} + G_2^*\delta_0^2 + G_3\delta_{-\frac{1}{2}}^3 + G_4^*\delta_0^4 + G_5\delta_{-\frac{1}{2}}^5 \cdots \qquad 0 \le p \le 1$$

In the above, $G_{2n} = \binom{p+n-1}{2n}$

$$G_{2n}^* = \binom{p+n}{2n}$$

$$G_{2n+1} = \binom{p+n}{2n+1}$$

Stirling's Formula

$$f_p = f_0 + \tfrac{1}{2}p\left(\delta_{\frac{1}{2}} + \delta_{-\frac{1}{2}}\right) + \tfrac{1}{2}p^2\delta_0^2 + S_3\left(\delta_{\frac{1}{2}}^3 + \delta_{-\frac{1}{2}}^3\right) + S_4\delta_0^4 + \cdots \qquad -\tfrac{1}{2} \le p \le \tfrac{1}{2}$$

Steffenson's Formula

$$f_p = f_0 + \tfrac{1}{2}p(p+1)\delta_{\frac{1}{2}} - \tfrac{1}{2}(p-1)p\delta_{-\frac{1}{2}} + (S_3 + S_4)\delta_{\frac{1}{2}}^3 + (S_3 - S_4)\delta_{-\frac{1}{2}}^3 \cdots \qquad -\tfrac{1}{2} \le p \le \tfrac{1}{2}$$

© 2003 by CRC Press LLC

$$\text{In the above, } S_{2n+1} = \frac{1}{2}\binom{p+n}{2n+1}$$

$$S_{2n+2} = \frac{p}{2n+2}\binom{p+n}{2n+1}$$

$$S_{2n+1} + S_{2n+2} = \binom{p+n+1}{2n+2}$$

$$S_{2n+1} - S_{2n+2} = -\binom{p+n}{2n+2}$$

Bessel's Formula

$$f_p = f_0 + p\delta_{\frac{1}{2}} + B_2\left(\delta_0^2 + \delta_1^2\right) + B_3\delta_{\frac{1}{2}}^3 + B_4(\delta_0^4 + \delta_1^4) + B_5\delta_{\frac{1}{2}}^5 + \cdots \qquad 0 \le p \le 1$$

Everett's Formula

$$f_p = (1-p)f_0 + pf_1 + E_2\delta_0^2 + F_2\delta_1^2 + E_4\delta_0^4 + F_4\delta_1^4 + E_6\delta_0^6 + F_6\delta_1^6 + \cdots \qquad 0 \le p \le 1$$

The coefficients in the above two formulae are related to each other and to the coefficients in the Gaussian formulae by the identities

$$\begin{aligned}
B_{2n} &\equiv \tfrac{1}{2}G_{2n} \equiv \tfrac{1}{2}(E_{2n} + F_{2n}) \\
B_{2n+1} &\equiv G_{2n+1} - \tfrac{1}{2}G_{2n} \equiv \tfrac{1}{2}(F_{2n} - E_{2n}) \\
E_{2n} &\equiv G_{2n} - G_{2n+1} \equiv B_{2n} - B_{2n+1} \\
F_{2n} &\equiv G_{2n+1} \equiv B_{2n} + B_{2n+1}
\end{aligned}$$

Also, for $q \equiv 1 - p$ the following symmetrical relationships hold:

$$\begin{aligned}
B_{2n}(p) &\equiv B_{2n}(q) \\
B_{2n+1}(p) &\equiv -B_{2n+1}(q) \\
E_{2n}(p) &\equiv F_{2n}(q) \\
F_{2n}(p) &\equiv E_{2n}(q)
\end{aligned}$$

as can be seen from the tables of these coefficients.

Bessel's Formula (Unmodified)

$$f_p = f_0 + p\delta_{\frac{1}{2}} + B_2\left(\delta_0^2 + \delta_1^2\right) + B_3\delta_{\frac{1}{2}}^3 + B_4(\delta_0^4 + \delta_1^4) + B_5\delta_{\frac{1}{2}}^5 + B_6\left(\delta_0^6 + \delta_1^6\right) + B_7\delta_{\frac{1}{2}}^7 + \cdots$$

© 2003 by CRC Press LLC

Lagrange's Interpolation Formula

$$f(x) = \frac{(x-x_1)(x-x_2)\ldots(x-x_n)}{(x_0-x_1)(x_0-x_2)\ldots(x_0-x_n)} f(x_0)$$

$$+ \frac{(x-x_0)(x-x_2)\ldots(x-x_n)}{(x_1-x_0)(x_1-x_2)\ldots(x_1-x_n)} f(x_1)$$

$$+ \cdots + \frac{(x-x_0)(x-x_1)\ldots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\ldots(x_n-x_{n-1})} f(x_n)$$

Newton's Divided Difference Formula

$$\begin{aligned} f(x) &= f_0 + (x + x_0) f[x_0, x_1] + (x-x_0)(x-x_1) f[x_0, x_1, x_2] \\ &\quad + \cdots + (x-x_0)(x-x_1)\ldots(x-x_{n-1}) f[x_0, x_1, \ldots, x_n] \end{aligned}$$

where

$$f[x_0, x_1] = \frac{f_1 - f_0}{x_1 - x_0}$$

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$$

$$f[x_0, x_1, \ldots, x_k] = \frac{f[x_1, x_2, \ldots, x_k] - f[x_0, x_1, \ldots, x_{k-1}]}{x_k - x_0}$$

The layout of a divided difference table is similar to that of an ordinary finite difference table.

$$\begin{array}{cccccc}
x_{-1} & f_{-1} & & \Delta^2_{-1} & & \Delta^4_{-1} \\
 & & \Delta_{-\frac{1}{2}} & & \Delta_{-\frac{1}{2}} & \\
x_0 & f_0 & & \Delta^2_0 & & \Delta^4_0 \\
 & & \Delta_{\frac{1}{2}} & & \Delta^3_{\frac{1}{2}} & \\
x_1 & f_1 & & \Delta^2_1 & & \Delta^4_1
\end{array}$$

where the Δ's are defined as follows:

$$\Delta^0_r \equiv f_r, \qquad \Delta_{r+\frac{1}{2}} \equiv (f_{r+1} - f_r)/(x_{r+1} - x_r)$$

and in general

$$\Delta^{2n}_r \equiv \left(\Delta^{2n-1}_{r+\frac{1}{2}} - \Delta^{2n-1}_{r-\frac{1}{2}}\right)/(x_{r+n} - x_{r-n})$$

and

$$\Delta^{2n+1}_{r+\frac{1}{2}} \equiv (\Delta^{2n}_{r+1} - \Delta^{2n}_r)/(x_{r+1+n} - x_{r-n})$$

© 2003 by CRC Press LLC

Iterative Linear Interpolation

Neville's modification of Aiken's method of iterative linear interpolation is one of the most powerful methods of interpolation when the arguments are unevenly spaced, as no prior knowledge of the order of the approximating polynomial is necessary nor is a difference table required.

The values obtained are successive approximations to the required result and the process terminates when there is no appreciable change. These values are, of course, useless if a new interpolation is required when the procedure must be started afresh.

Defining

$$f_{r,s} \equiv \frac{(x_s - x)f_r - (x_r - x)f_s}{(x_s - x_r)}$$

$$f_{r,s,t} \equiv \frac{(x_t - x)f_{r,s} - (x_r - x)f_{s,t}}{(x_t - x_r)}$$

$$f_{r,s,t,u} \equiv \frac{(x_u - x)f_{r,s,t} - (x_t - x)f_{s,t,u}}{(x_u - x_r)}$$

the computation is laid out as follows:

$$\begin{array}{ccccccc}
x_{-1} & (x_{-1} - x) & f_{-1} & & & \\
 & & & f_{-1,0} & & \\
x_0 & (x_0 - x) & f_0 & & f_{-1,0,1} & \\
 & & & f_{0,1} & & f_{-1,0,1,2} \\
x_1 & (x_1 - x) & f_1 & & f_{0,1,2} & \\
 & & & f_{1,2} & & \\
x_2 & (x_2 - x) & f_2 & & &
\end{array}$$

As the iterates tend to their limit, the common leading figures can be omitted.

Gauss's Trigonometric Interpolation Formula

This is of greatest value when the function is periodic, i.e., a Fourier series expansion is possible.

$$f(x) = \sum_{r=0}^{n} C_r f_r$$

where $C_r = N_r(x)/N_r(x_r)$ and

$$N_r(x) = \left[\sin\frac{(x - x_0)}{2}\right]\left[\sin\frac{(x - x_1)}{2}\right]\cdots\left[\sin\frac{(x - x_{r-1})}{2}\right]\left[\sin\frac{(x - x_{r+1})}{2}\right]\cdots\left[\sin\frac{(x - x_n)}{2}\right]$$

This is similar to the Lagrangian formula.

Reciprocal Differences

These are used when the quotient of two polynomials will give a better representation of the interpolating function than a simple polynomial expression.

© 2003 by CRC Press LLC

A convenient layout is as shown below:

$$
\begin{array}{cccccc}
x_{-1} & f_{-1} & & & & \\
 & & \rho_{-\frac{1}{2}} & & & \\
x_0 & f_0 & & \rho_0^2 & & \\
 & & \rho_{\frac{1}{2}} & & \rho_{\frac{1}{2}}^3 & \\
x_1 & f_1 & & \rho_1^2 & & \rho_1^4 \\
 & & \rho_{1\frac{1}{2}} & & \rho_{1\frac{1}{2}}^3 & \\
x_2 & f_2 & & \rho_2^2 & & \\
 & & \rho_{2\frac{1}{2}} & & & \\
x_3 & f_3 & & & &
\end{array}
$$

where
$$\rho_{r+\frac{1}{2}} \equiv \frac{x_{r+1} - x_r}{f_{r+1} - f_r}$$

and
$$\rho_r^2 \equiv \frac{x_{r+1} - x_{r-1}}{f_{r+\frac{1}{2}} - f_{r-\frac{1}{2}}} + f_r$$

In general,
$$\rho_{r+\frac{1}{2}}^{2n+1} \equiv \frac{x_{r+n+1} - x_{r-n}}{\rho_{r+1}^{2n} - \rho_r^{2n}} + \rho_{r+\frac{1}{2}}^{2n-1}$$

$$\rho_r^{2n} \equiv \frac{x_{r+n} - x_{r-n}}{\rho_{r+1}^{2n-1} - \rho_{r-\frac{1}{2}}^{2n-1}} + \rho_r^{2n-2}$$

The interpolation formula is expressed in the form of a continued fraction expansion.

The expansion corresponding to Newton's forward difference interpolation formula, in the sense of the differences involved, is

$$f(x) = f_0 + \cfrac{(x - x_0)}{\rho_{\frac{1}{2}} + \cfrac{(x_2 - x_1)}{\rho_1 - f_0 + \cfrac{(x - x_2)}{\rho_{1\frac{1}{2}}^3 - \rho_{\frac{1}{2}} + \cfrac{(x_4 - x_3)}{\rho_2^4 - \rho_1^2 + \cfrac{(x - x_4)}{\text{etc.}}}}}}$$

while that corresponding to Gauss' forward formula is

© 2003 by CRC Press LLC

$$f(x) = f_0 + \cfrac{(x - x_0)}{\rho_{\frac{1}{2}} + \cfrac{(x_2 - x_1)}{\rho_0^2 - f_0 + \cfrac{(x_3 - x_{-1})}{\rho_{\frac{1}{2}}^3 - \rho_{\frac{1}{2}} + \cfrac{(x_4 - x_2)}{\rho_0^4 - \rho_0^2 + \cfrac{(x - x_{-2})}{\text{etc.}}}}}}$$

Probability

Definitions

A sample space S associated with an experiment is a set S of elements such that any outcome of the experiment corresponds to one and only one element of the set. An event E is a subset of a sample space S. An element in a sample space is called a sample point or a simple event (unit subset of S).

Definition of Probability

If an experiment can occur in n mutually exclusive and equally likely ways, and if exactly m of these ways correspond to an event E, then the probability of E is given by

$$P(E) = \frac{m}{n}$$

If E is a subset of S, and if to each unit subset of S a non-negative number, called its probability, is assigned, and if E is the union of two or more different simple events, then the probability of E, denoted by $P(E)$, is the sum of the probabilities of those simple events whose union is E.

Marginal and Conditional Probability

Suppose a sample space S is partitioned into rs disjoint subsets where the general subset is denoted by $E_i \cap F_j$. Then the marginal probability of E_i is defined as

$$P(E_i) = \sum_{j=1}^{s} P(E_i \cap F_j)$$

and the marginal probability of F_j is defined as

$$P(F_j) = \sum_{i=1}^{r} P(E_i \cap F_j)$$

The conditional probability of E_i, given that F_j has occurred, is defined as

$$P(E_i/F_j) = \frac{P(E_i \cap F_j)}{P(F_j)}, \qquad P(F_j) \neq 0$$

and that of F_j, given that E_i has occurred, is defined as

$$P(F_j/E_i) = \frac{P(E_i \cap F_j)}{P(E_i)}, \qquad P(E_i) \neq 0$$

© 2003 by CRC Press LLC

Probability Theorems

1. If ϕ is the null set, $P(\phi) = 0$.
2. If S is the sample space, $P(S) = 1$.
3. If E and F are two events,

$$P(E \cup F) = P(E) + P(F) - P(E \cap F)$$

4. If E and F are mutually exclusive events,

$$P(E \cup F) = P(E) + P(F)$$

5. If E and E' are complementary events,

$$P(E) = 1 - P(E')$$

6. The conditional probability of an event E, given an event F, is denoted by $P(E/F)$ and is defined as

$$P(E/F) = \frac{P(E \cap F)}{P(F)}$$

where $P(F) \neq 0$.
7. Two events E and F are said to be independent if and only if

$$P(E \cap F) = P(E) \cdot P(F)$$

E is said to be statistically independent of F if $P(E/F) = P(E)$ and $P(F/E) = P(F)$.
8. The events $E_1, E_2, \ldots, E_n$ are called mutually independent for all combinations if and only if every combination of these events taken any number at a time is independent.
9. *Bayes Theorem.*
If $E_1, E_2, \ldots, E_n$ are n mutually exclusive events whose union is the sample space S, and E is any arbitrary event of S such that $P(E) \neq 0$, then

$$P(E_k/E) = \frac{P(E_k) \cdot P(E/E_k)}{\sum_{j=1}^{n} [P(E_j) \cdot P(E/E_j)]}$$

Random Variable

A function whose domain is a sample space S and whose range is some set of real numbers is called a random variable, denoted by **X.** The function **X** transforms sample points of S into points on the x-axis. **X** will be called a discrete random variable if it is a random variable that assumes only a finite or denumerable number of values on the x-axis. **X** will be called a continuous random variable if it assumes a continuum of values on the x-axis.

Probability Function (Discrete Case)

The random variable **X** will be called a discrete random variable if there exists a function f such that $f(x_i) \geq 0$ and $\sum_i f(x_i) = 1$ for $i = 1, 2, 3, \ldots$ and such that for any event E,

$$P(E) = P[\mathbf{X} \text{ is in } E] = \sum_E f(x)$$

where $\sum_E$ means sum $f(x)$ over those values x_i that are in E and where $f(x) = P[\mathbf{X} = x]$.

© 2003 by CRC Press LLC

The probability that the value of **X** is some real number x is given by $f(x) = P[\mathbf{X} = x]$, where f is called the probability function of the random variable **X.**

Cumulative Distribution Function (Discrete Case)

The probability that the value of a random variable **X** is less than or equal to some real number x is defined as

$$\begin{aligned} F(x) &= P(\mathbf{X} \le x) \\ &= \sum f(x_i), \qquad -\infty < x < \infty \end{aligned}$$

where the summation extends over those values of i such that $x_i \le x$.

Probability Density (Continuous Case)

The random variable **X** will be called a continuous random variable if there exists a function f such that $f(x) \ge 0$ and $\int_{-\infty}^{\infty} f(x)\,dx = 1$ for all x in interval $-\infty < x < \infty$ and such that, for any event E,

$$P(E) = P(\mathbf{X} \text{ is in } E) = \int_E f(x)\,dx$$

$f(x)$ is called the probability density of the random variable **X.** The probability that **X** assumes any given value of x is equal to zero, and the probability that it assumes a value on the interval from a to b, including or excluding either endpoint, is equal to

$$\int_a^b f(x)\,dx$$

Cumulative Distribution Function (Continuous Case)

The probability that the value of a random variable **X** is less than or equal to some real number x is defined as

$$\begin{aligned} F(x) &= P(\mathbf{X} \le x), \qquad -\infty < x < \infty \\ &= \int_{-\infty}^{\infty} f(x)\,dx. \end{aligned}$$

From the cumulative distribution, the density, if it exists, can be found from

$$f(x) = \frac{dF(x)}{dx}$$

From the cumulative distribution

$$\begin{aligned} P(a \le \mathbf{X} \le b) &= P(\mathbf{X} \le b) - P(\mathbf{X} \le a) \\ &= F(b) - F(a) \end{aligned}$$

Mathematical Expectation

Expected Value

Let **X** be a random variable with density $f(x)$. Then the expected value of **X,** $E(\mathbf{X})$, is defined to be

$$E(\mathbf{X}) = \sum_x x f(x)$$

© 2003 by CRC Press LLC

if **X** is discrete and

$$E(\mathbf{X}) = \int_{-\infty}^{\infty} x f(x)\, dx$$

if **X** is continuous. The expected value of a function g of a random variable **X** is defined as

$$E[g(\mathbf{X})] = \sum_x g(x) \cdot f(x)$$

if **X** is discrete and

$$E[g(\mathbf{X})] = \int_{-\infty}^{\infty} g(x) \cdot f(x) dx$$

if **X** is continuous.

Positional Notation

In our ordinary system of writing numbers, the value of any digit depends on its position in the number. The value of a digit in any position is ten times the value of the same digit one position to the right, or one-tenth the value of the same digit one position to the left. Thus, for example,

$$173.246 = 1 \times 10^2 + 7 \times 10^1 + 3 + 2 \times \frac{1}{10} + 4 \times \frac{1}{10^2} + 6 \times \frac{1}{10^3}$$

There is no reason that a number other than 10 cannot be used as the *base,* or *radix,* of the number system. In fact, bases of 2, 8, and 16 are commonly used in working with digital computers. When the base used is not clear from the context, it is usually indicated as a parenthesized subscript or merely as a subscript. Thus,

$$743_{(8)} = 7 \times 8^2 + 4 \times 8 + 3 = 7 \times 64 + 4 \times 8 + 3 = 448 + 32 + 3 = 483_{(10)}$$

$$1011.101_{(2)} = 1 \times 2^3 + 0 \times 2^2 + 1 \times 2 + 1 + 1 \times \frac{1}{2} + 0 \times \frac{1}{4} + 1 \times \frac{1}{8} = 11.625_{(10)}$$

Change of Base

In this section, it is assumed that all calculations will be performed in base 10, since this is the only base in which most people can easily compute. However, there is no logical reason that some other base could not be used for the computations.

To convert a number from another base into base 10:

Simply write down the digits of the number, with each one multiplied by its appropriate positional value. Then perform the indicated computations in base 10, and write down the answer.

For examples, see the two examples in the previous section.

To convert a number from base 10 into another base:

The part of the number to the left of the point and the part to the right must be operated on separately. For the integer part (the part to the left of the point):

a. Divide the number by the new base, getting an integer quotient and remainder.
b. Write down the remainder as the last digit of the number in the new base.
c. Using the quotient from the last division in place of the original number, repeat the above two steps until the quotient becomes zero.

© 2003 by CRC Press LLC

For the fractional part (the part to the right of the point):

a. Multiply the number by the new base.
b. Write down the integral part of the product as the first digit of the fractional part in the new base.
c. Using the fractional part of the last product in place of the original number, repeat the above two steps until the product becomes an integer or until the desired number of places have been computed.

Examples

These examples show a convenient method of arranging the computations.

1. Convert $103.118_{(10)}$ to base 8.

8	|103|	7	
8	|12|	4	
		1	147.074324...

The calculation of the fractional part could be carried out as far as desired.
It is a non-terminating fraction that will eventually repeat itself.

$$103.118_{(10)} = 147.074324\ldots_{(8)}$$

The calculations may be further shortened by not writing down the multiplier and divisor at each step of the algorithm, as shown in the next example.

2. Convert $275.824_{(10)}$ to base 5.

5	|275|	0	.824
	|55|	0	4.120
	|11|	1	0.600
	2		3.000

$$275.824_{(10)} = 2100.403_{(5)}$$

To convert from one base to another (neither of which is 10):

The easiest procedure is usually to convert first to base 10 and then to the desired base. However, there are two exceptions to this:

1. If a computational facility is possessed in either of the bases, it may be used instead of base 10, and the appropriate one of the above methods may be applied.
2. If the two bases are different powers of the same number, the conversion may be done digit-by-digit to the base that is the common root of both bases and then digit-by-digit back to the other base.

   Example: Convert $127.653_{(8)}$ to base 16. (For base 16, the letters A–F are used for the digits $10_{(10)}$–$15_{(10)}$.)

The first step is to convert the number to base 2, simply by converting each digit to its binary equivalent:

$$127.653_{(8)} = 001\ 010\ 111\ \cdot\ 110\ 101\ 011_{(2)}$$

© 2003 by CRC Press LLC

Now by simply regrouping the binary number into groups of four binary digits, starting at the point, we convert to base 16:

$$127.653_{(8)} = 101\ 0111\ \cdot\ 1101\ 0101\ 1_{(2)} = 57.D58_{(16)}$$

Credits

Material in this section was reprinted from the following sources:

D.R. Lide, Ed., *CRC Handbook of Chemistry and Physics,* 73rd ed., Boca Raton, FL: CRC Press. 1992: International System of Units (SI), conversion constants and multipliers (conversion of temperatures), symbols and terminology for physical and chemical quantities, fundamental physical constants.

W.H. Beyer, Ed., *CRC Standard Mathematical Tables and Formulae,* 29th ed., Boca Raton, FL: CRC Press, 1991: Greek alphabet, conversion constants and multipliers (recommended decimal multiples and submultiples, metric to English, English to metric, general, temperature factors), physical constants, series expansion, integrals, the Fourier transforms, numerical methods, probability, positional notation.

R.J. Tallarida, *Pocket Book of Integrals and Mathematical Formulas,* 2nd ed., Boca Raton, FL: CRC Press, 1992: Elementary algebra and geometry; determinants, matrices, and linear systems of equations; trigonometry; analytic geometry; series; differential calculus; integral calculus; vector analysis; special functions; statistics; tables of probability and statistics; table of derivatives.

Associations and Societies

American Concrete Institute (ACI)
PO Box 9094
Farmington Hills, MI 48333
Tel. # (248) 848-3700
Homepage: http://www.aci-int.net/

Founded in 1905, the American Concrete Institute (ACI) has grown into a chartered society with over 20,000 members worldwide. The ACI is a technical and educational nonprofit society dedicated to improving the design, construction, manufacture, and maintenance of concrete structures.

Among ACI's 20,000 members are structural designers, architects, civil engineers, educators, contractors, concrete craftsmen and technicians, representatives of materials suppliers, students, testing laboratories, and manufacturers from around the world. The 83 national and international chapters provide the membership with opportunities to network with their peers and keep in tune with the activities of ACI International.

Membership

Membership is open to individuals who work directly in, have an association with, or have an interest in concrete. All members are encouraged to participate in the activities of the ACI International, which include involvement on voluntary technical committees that develop ACI codes, standards, and reports. Various levels of membership exist to meet particular needs. Student memberships are available.

Publications

Concrete International. Published monthly. Covers institute, chapter, and industry news. Several technical articles following a specific theme appear in each issue.

ACI Materials Journal. Published bimonthly. Describes research in materials and concrete, related ACI International standards, and committee reports.

© 2003 by CRC Press LLC

ACI Structural Journal. Published bimonthly. Includes technical papers on structural design and analysis, state-of-the-art reviews on reinforced and structural elements, and the use and handling of concrete.

Other publications: ACI International makes available over 300 technical publication on concrete. Information is also available in computer software and compact disc formats. A free 72-page publications catalog describing what ACI International has to offer is available.

Other Activities

ACI International provides technical information in the form of high-quality conventions, seminars, and symposia.

American Iron and Steel Institute (AISI)
1101 17th Street NW, Suite 1300
Washington, DC 20036
Tel. (202) 452-7100
Homepage: http://www.steel.org/

The American Iron and Steel Institute (AISI) was founded in 1908. The institute is a nonprofit association of North American companies engaged in the iron and steel industry. AISI comprises 43 member companies that produce the full range of steel mill products. Also included are iron ore mining companies and member companies that produce raw steel, including integrated, electric furnace, and reconstituted mills. Member companies account for more than two-thirds of the raw steel produced in the U.S., most of the steel manufactured in Canada, and nearly two-thirds of the flat-rolled steel products manufactured in Mexico.

AISI has 230 associate members, including customers who distribute, fabricate, process, or consume steel. Also included are companies and representatives of organizations that supply the steel industry with materials, equipment, and services, as well as individuals associated with educational or research organizations.

American National Standards Institute (ANSI)
Washington, DC, Headquarters
1819 L Street NW, 6th Fl.
Washington, DC 20036
Tel. (202) 293-8020
Fax. (202) 293-9287

New York City Office
25 West 43rd Street, 4th Floor
New York, NY, 10036
Tel. (212) 642-4900
Fax. (212) 398-0023
Homepage: http://www.ansi.org/
E-mail: info@ansi.org

Founded in 1918, the American National Standards Institute (ANSI) is a private, nonprofit membership organization that coordinates the U.S. voluntary consensus standards system and approves American National Standards. ANSI ensures that a single set of nonconflicting American National Standards are developed by ANSI-accredited standards developers and that all interests concerned have the opportunity to participate in the development process.

ANSI is the official U.S. representative to the International Accreditation Forum (IAF), the International Organization for Standardization (ISO), and, via the U.S. National Committee, the International Electrotechnical Commission (IEC). ANSI is also the U.S. member of the Pacific Area Standards Congress (PASC) and the Pan American Standards Commission (COPANT).

© 2003 by CRC Press LLC

Membership

ANSI consists of approximately 1300 national and international companies, 30 government agencies, 20 institutional members, and 250 professional, technical, trade, labor, and consumer organizations. ANSI offers no individual membership. For more information on membership, write to the Member Services Department at the New York Office; call (212) 642-4900; or e-mail membership@ansi.org.

Publications

ANSI Reporter. Published monthly. Newsletter that updates members on major national and international standards activities. It also provides information on the activities of the European standards bodies, CEN and CENELEC.

Standards Action. Published biweekly. This newsletter outlines all national draft standards currently under consideration for approval as American National Standards and solicits comments from readers. Comments are also solicited on regional, international, and foreign standards. These comments are then reviewed as part of the development process.

Catalog of American National Standards. Published annually. Provides a complete listing of all ANSI-approved American National Standards. Supplements are also published.

American Railway Engineering and Maintenance-of-Way Association (AREMA)
8201 Corporate Drive, Suite 1125
Landover, MD 20785
Tel. (301) 459-3200
Fax. (301) 459-8077
Homepage: http://www.arema.org

The American Railway Engineering and Maintenance-of-Way Association (AREMA) was formed on October 1, 1997, as the result of a merger of three engineering support associations, namely the American Railway Bridge and Building Association, the American Railway Engineering Association, and the Roadmasters and Maintenance of Way Association, along with functions of the Communications and Signal Division of the Association of American Railroads. The rich history of the predecessor organizations, each having over 100 years of service to the rail industry, is the legacy of AREMA.

Each of the four groups — Roadmasters and Maintenance of Way Association, American Railway Bridge and Building Association, American Railway Engineering Association, and Communications and Signal Division — that came together to form AREMA have, in their own way, built an excellent foundation upon which to base the new association, whose mission is the development and advancement of both technical and practical knowledge and recommended practices pertaining to the design, construction, and maintenance of railway infrastructure.

Membership

The basic qualifications for membership are five years of experience in the profession of maintaining, operating, constructing, or locating railways. Graduation from a recognized college or university with a degree in engineering is being taken as the equivalent of three years of experience.

Publications

AREMA Manual for Railway Engineering comprises the work of the association's committees. The manual is revised annually to make the latest in recommended practice information for railway engineering available to all interested parties. The *Portfolio of Trackwork Plans* is also compiled and updated in the same manner.

American Society of Civil Engineers (ASCE)
International Headquarters
1801 Alexander Bell Drive
Reston, VA 20191-4400

© 2003 by CRC Press LLC

Tel. 1-800-548-2723 (toll-free) / (703) 295-6300
Fax. (703) 295-6222 / (703) 295-6444 (fax-back)

Washington Office
1015 15th Street NW, Suite 600
Washington, DC 20005
Tel. (202) 789-2200
Fax. (202) 289-6797

Founded in 1852, the American Society of Civil Engineers (ASCE) is America's oldest national professional engineering society. The society has more than 115,000 individual members, including 6,500 international members in 137 nations. Memberships consist of individual professional engineers rather than companies or organizations.

ASCE is organized geographically into 21 district councils, 83 sections, 143 branches, and 246 student chapters and clubs. The society is governed by a 28-member board and is headquartered in the United Engineering Center in New York City. A Washington, DC, office is maintained for government relations.

ASCE maintains the Civil Engineering Research Foundation to focus national attention and resources on the research needs of the civil engineering profession. In addition, there are 25 technical divisions and councils that foster the development and advancement of the science and practice of engineering. ASCE has marked infrastructure renewal as a top national priority.

ASCE is the world's largest publisher of civil engineering information, publishing over 63,000 pages in 1994. Nearly 42% of the society's yearly income is generated through publication sales.

Membership

Membership applicants must meet the requirements set in the constitution of the ASCE. Various levels of membership exist to meet particular needs. Student memberships are available to students who meet the requirements of the constitution. Various entrance fees and dues are required of the various levels of membership. Application materials may be requested by mailing the ASCE Membership Services Department, phoning 800-548-2723 (toll-free in the United States) or 703-295-6300 (internationally), faxing 703-295-6333, or e-mailing your request to memapp@asce.org.

Publications

Civil Engineering. Published monthly. This is the society's official magazine and is mailed to all members of ASCE. The magazine contains articles of current interest in the various fields of civil engineering, news of a professional nature, and reports on the activities of ASCE and its members. Independently prepared papers may be sent directly to the editor of *Civil Engineering* at 345 East 47th Street, New York, NY 10017-2398.

ASCE News. Published monthly. Mailed to all members without charge. It concentrates on the activities of ASCE and its members, with the intent of promoting interest and participation in society programs.

Worldwide Projects. Published quarterly. A copublication of ASCE and Intercontinental Media, Inc., Westport, CT. Each issue provides engineers with articles giving insight into various topics related to international civil engineering projects and doing business outside the U.S.

Journals published: *Journal of Management in Engineering*, published bimonthly, and *Journal of Professional Issues in Engineering Education and Practice*, published quarterly, present professional and technical problems of broad interest and implications. ASCE also publishes significant reports of the Professional Activities Committee and its constituent committees.

Other publications: The society also publishes transactions; standards; engineer-, owner-, and construction-related documents; the publications information and indexes; and newsletters. A civil engineering database is also available. For inquiries on prices or to request a catalog or sample issues, e-mail marketing@asce.org; phone 1-800-548-2723, ext. 6251 (U.S.), or 703-295-6163 (international); fax 703-295-6278; or mail American Society of Civil Engineers, Publications Marketing Department, 1801 Alexander Bell Drive, Reston, VA 20191-4400.

© 2003 by CRC Press LLC

American Society for Testing and Materials (ASTM)
International Headquarters
100 Barr Harbor Drive
West Conshohocken, PA 19428-2959
Tel. (610) 832-9500
Fax (610) 832-9555
Homepage: http://www.astm.org/

Founded in 1898, the American Society for Testing and Materials (ASTM) has grown into one of the largest voluntary standards development systems in the world. ASTM is a nonprofit organization that provides a forum for producers, users, ultimate consumers, and those having a general interest, such as representatives of government and academia, to meet on common ground and write standards for materials, products, systems, and services. From the work of 131 standard-writing committees, ASTM publishes standard test methods, specifications, practices, guides, classifications, and terminology. ASTM's standards development activities encompass metals, paints, plastics, textiles, petroleum, construction, energy, the environment, consumer products, medical services and devices, computerized systems, electronics, and many other areas. All technical research and testing are done voluntarily by more than 35,000 technically qualified ASTM members located throughout the world.

Membership

ASTM members pay an annual administrative fee of $75 for individual membership and $400 for an organizational membership. The only other costs involved are the time and travel expenses of the committee members and the donated use of members' laboratory and research facilities.

Publications

Annual Book of ASTM Standards. A 70-volume set that includes standards and specs in the following subject areas:

Iron and steel products
Nonferrous metal products
Metals test methods and analytical procedures
Construction
Petroleum products, lubricants, and fossil fuels
Medical devices and services
General methods and instrumentation
Paints, related coatings, and aromatics
Textiles
Plastics
Rubber
Electrical insulation and electronics
Water and environmental technology
Nuclear, solar, and geothermal energy
General products, chemical specialties, and end-use products

Discounts are applied when purchased as a complete set or when purchased by complete sections. Volumes may also be purchased individually.

Standardization News. Published monthly.

Journals published: *Journal of Testing and Evaluation*; *Cement, Concrete, and Aggregates*; *Geotechnical Testing Journal*, *Journal of Composites Technology and Research*; and *Journal of Forensic Sciences*.

ASTM also publishes books containing reports on state-of-the-art testing techniques and their possible applications.

© 2003 by CRC Press LLC

American Water Works Association
Headquarters
6666 West Quincy Avenue
Denver, CO 80235
Tel. (303) 794-7711
Fax. (303) 794-7310

Government Affairs Office
1401 New York Avenue NW, Suite 640
Washington, DC 20005
Tel. (202) 628-8303
Fax. (202) 628-2846
Homepage: http://www.awwa.org/

The American Water Works Association (AWWA) was established in 1881 by 22 dedicated water supply professionals. Membership has grown to more than 54,000 individuals and organizations. AWWA is an international, nonprofit, scientific, and educational association dedicated to improving drinking water for people everywhere. Today, AWWA has grown to be the largest organization of water supply professionals in the world, boasting members from virtually every country.

AWWA was formed to promote public health, safety, and welfare through the improvement of the quality and quantity of water delivered to the public and through the development of public understanding. AWWA also takes an active role in shaping the water industry's direction through research, participation in legislative activities, development of products, procedural standards, and manuals of practice, and it educates the public on water issues to promote a spirit of cooperation between consumers and buyers.

Membership

Listed under individual memberships are active, affiliate, and student. Organization memberships include utility, municipal service subscriber, small water system, associate, consultant, contractor, technical service, and manufacturer's agent, distributor, or representative. The association is governed by a board of directors that establishes policy for the overall management and direction of association affairs.

Publications

AWWA is the world's major publisher of drinking water information. Its publications cover just about every area of interest in the water supply field. More than 500 titles are offered, covering all aspects of water resources, water quality, treatment and distribution, utility management, and employee training and safety.

Civil Engineering Research Foundation (CERF)
2131 K Street NW, Suite 700
Washington, DC 20037
Tel. (202) 785-6420
Fax. (202) 833-2604
Homepage: http://www.cerf.org/

The Civil Engineering Research Foundation (CERF) was created by the American Society of Civil Engineers and began operation in 1989 to advance the civil engineering profession through research. CERF is an industry-guided research organization that serves as a critical catalyst to help the design and construction industry and the civil engineering profession expedite the transfer of research results into practice through cooperative national programs. CERF integrates the efforts of industry, government, and academia in order to implement research that is beyond the capabilities of any single organization. CERF is an independent, nonprofit organization but remains affiliated with ASCE.

© 2003 by CRC Press LLC

Council on Tall Buildings and Urban Habitat
Lehigh University
117 ATLSS Drive
Bethlehem, PA 18015
Tel. (215) 758-3515
Fax (215) 758-4522
Homepage: http://www.lehigh.edu/~inctbuh/
E-mail: inctbuh@lehigh.edu

The Council on Tall Buildings and Urban Habitat is an international organization sponsored by engineering, architectural, and planning professionals. The council was founded in 1969 and was known as the Joint Committee on Tall Buildings until the name was changed in 1976 to its present form.

The council was established to study and report on all aspects of the planning, design, construction, and operation of tall buildings. The council is also concerned with the role of tall buildings in the urban environment and their impact thereon. However, the council is not an advocate for tall buildings per se, but in those situations in which they are viable, the council seeks to encourage the use of the latest knowledge in their implementation.

Membership

Membership is available to associations, commercial organizations, individual members, and students. Membership is available to students at the rate of $10 per year. Membership fees vary for associations, commercial organizations, and individuals.

Publications

A major focus of the council is the publication of a comprehensive monograph series for use by those responsible for tall building planning and design. The original five-volume *Monograph on the Planning and Design of Tall Buildings* was released between 1978 and 1981. This comprehensive source of tall building information is the only such reference tool now available to the high-rise specialist. The volumes are *Planning and Environmental Criteria for Tall Buildings*, *Tall Building Systems and Concepts*, *Tall Building Criteria and Loading*, *Structural Design of Tall Street Buildings*, and *Structural Design of Tall Concrete and Masonry Buildings*. These volumes are available as a set or sold separately. Updated monographs are continually added to the series in order to keep information current.

Structural Stability Research Council
Headquarters
University of Florida
Department of Civil and Coastal Engineering
345 Weil Hall, PO Box 116580
Gainesville, FL 32611-6580
Tel. (352) 846-3874, ext. 1424
Fax. (352) 846-3978
Homepage: http://www.ce.ufl.edu/~ssrc/
Email: ssrc@ce.ufl.edu

The Structural Stability Research Council (formerly the Column Research Council) was founded in 1944 to review and resolve the conflicting opinions and practices that existed at the time with respect to solutions to stability problems and to facilitate and promote economical and safe design. Now, more than 50 years later, the council has broadened its scope within the field of structural stability, has become international in character, and continues to seek solutions to stability problems.

© 2003 by CRC Press LLC

Membership

Various levels of membership exist for individuals. Organizations, companies, and firms concerned with investigation and design of metal and composite structures are invited by the council to become sponsors, participating organizations, participating companies, or participating firms.

Publications

The council maintains a library at its headquarters. Material from the library is available on request.

Transportation Research Board (TRB)
Cecil and Ida Green Building
2001 Wisconsin Avenue NW
Washington, DC 20007
Tel. (202) 334-2934
Fax (202) 334-2003
Homepage: http://www.nas.edu/trb/

The Transportation Research Board (TRB) is a unit of the National Research Council, which serves the National Academy of Sciences and the National Academy of Engineering. The board's purpose is to stimulate research concerning the nature and performance of transportation systems, to disseminate the information produced by the research, and to encourage the application of appropriate research findings. The board's program is carried out by more than 330 committees, task forces, and panels composed of more than 3900 administrators, engineers, social scientists, attorneys, educators, and others concerned with transportation; they serve without compensation.

The program is supported by state transportation and highway departments, modal administrations of the U.S. Department of Transportation, and others interested in the development of transportation.

In November 1920, after a series of preliminary meetings and conferences, the National Research Council created the Advisory Board on Highway Research. Four years later, the name was changed to the Highway Research Board. During the late 1960s, the Highway Research Board expanded its scope to all modes of transportation. The name was again changed in 1974 to the Transportation Research Board to recognize its increased emphasis on a broadened approach to transportation problems and needs.

Today the Transportation Research Board devotes attention to all factors pertinent to the understanding, design, and function of systems for the safe and efficient movement of people and goods, including the following:

- Planning, design, construction, operation, safety, and maintenance of transportation facilities and their components
- Economics, financing, and administration of transportation facilities and services
- Interaction of transportation systems with one another and with the physical, economic, and social environment that they are designed to serve

Publications

One of the most important activities of the Transportation Research Board is the dissemination of current research results. The mainstay of the TRB publications program is the Transportation Research Record series. This series consists primarily of the papers delivered at the TRB annual meeting by authors from all over the world.

Ethics

The following code of ethics was adopted by the American Society of Civil Engineers on September 25, 1976. The code of ethics became effective on January 1, 1977. The ASCE has since amended this code on October 25, 1980, and April 17, 1993. The code of ethics shown below is in the most recent amended form.

© 2003 by CRC Press LLC

The ASCE adopted the fundamental principles of the ABET Code of Ethics of Engineers as accepted by the Accreditation Board for Engineering and Technology, Inc. (ABET).

Code of Ethics[1]

Fundamental Principles

Engineers uphold and advance the integrity, honor and dignity of the engineering profession by:

1. using their knowledge and skill for the enhancement of human welfare;
2. being honest and impartial and serving with fidelity the public, their employers and clients;
3. striving to increase the competence and prestige of the engineering profession; and
4. supporting the professional and technical societies of their disciplines.

Fundamental Canons

1. Engineers shall hold paramount the safety, health and welfare of the public in the performance of their professional duties.
2. Engineers shall perform services only in areas of their competence.
3. Engineers shall issue public statements only in an objective and truthful manner.
4. Engineers shall act in professional matters for each employer or client as faithful agents or trustees, and shall avoid conflicts of interest.
5. Engineers shall build their professional reputation on the merit of their services and shall not compete unfairly with others.
6. Engineers shall act in such a manner as to uphold and enhance the honor, integrity and dignity of the engineering profession.
7. Engineers shall continue their professional development throughout their careers, and shall provide opportunities for the professional development of those engineers under their supervision.

Guidelines to Practice Under the Fundamental Canons of Ethics

CANON 1. Engineers shall hold paramount the safety, health and welfare of the public in the performance of their professional duties.

a. Engineers shall recognize that the lives, safety, health and welfare of the general public are dependent upon engineering judgments, decisions and practices incorporated into structures, machines, products, processes and devices.
b. Engineers shall approve or seal only those design documents, reviewed or prepared by them, which are determined to be safe for public health and welfare in conformity with accepted engineering standards.
c. Engineers whose professional judgment is overruled under circumstances where the safety, health and welfare of the public are endangered, shall inform their clients or employers of the possible consequences.
d. Engineers who have knowledge or reason to believe that another person or firm may be in violation of any of the provisions of Canon 1 shall present such information to the proper authority in writing and shall cooperate with the proper authority in furnishing such further information or assistance as may be required.
e. Engineers should seek opportunities to be of constructive service in civic affairs and work for the advancement of the safety, health and well-being of their communities.
f. Engineers should be committed to improving the environment to enhance the quality of life.

[1]Published with permission of the American Society of Civil Engineers.

© 2003 by CRC Press LLC

CANON 2. Engineers shall perform services only in areas of their competence.

a. Engineers shall undertake to perform engineering assignments only when qualified by education or experience in the technical field of engineering involved.
b. Engineers may accept an assignment requiring education or experience outside of their own fields of competence, provided their services are restricted to those phases of the project in which they are qualified. All other phases of such project shall be performed by qualified associates, consultants or employees.
c. Engineers shall not affix their signatures or seals to any engineering plan or document dealing with subject matter in which they lack competence by virtue of education or experience or to any such plan or document not reviewed or prepared under their supervisory control.

CANON 3. Engineers shall issue public statements only in an objective and truthful manner.

a. Engineers should endeavor to extend the public knowledge of engineering, and shall not participate in the dissemination of untrue, unfair or exaggerated statements regarding engineering.
b. Engineers shall be objective and truthful in professional reports, statements or testimony. They shall include all relevant and pertinent information in such reports, statements or testimony.
c. Engineers, when serving as expert witnesses, shall express an engineering opinion only when it is founded upon adequate knowledge of the facts, upon a background of technical competence and upon honest conviction.
d. Engineers shall issue no statements, criticisms or arguments on engineering matters which are inspired or paid for by interested parties, unless they indicate on whose behalf the statements are made.
e. Engineers shall be dignified and modest in explaining their work and merit, and will avoid any act tending to promote their own interests at the expense of the integrity, honor and dignity of the profession.

CANON 4. Engineers shall act in professional matters for each employer or client as faithful agents or trustees, and shall avoid conflicts of interest.

a. Engineers shall avoid all known or potential conflicts of interest with their employers or clients and shall promptly inform their employers or clients of any business association, interests or circumstances which could influence their judgment or the quality of their services.
b. Engineers shall not accept compensation from more than one party for services on the same project, or for services pertaining to the same project, unless the circumstances are fully disclosed to and agreed to by all interested parties.
c. Engineers shall not solicit or accept gratuities, directly or indirectly, from contractors, their agents or other parties dealing with their clients or employers in connection with work for which they are responsible.
d. Engineers in public service as members, advisors or employees of a governmental body or department shall not participate in considerations or actions with respect to services solicited or provided by them or their organization in private or public engineering practice.
e. Engineers shall advise their employers or clients when, as a result of their studies, they believe a project will not be successful.
f. Engineers shall not use confidential information coming to them in the course of their assignments as a means of making personal profit if such action is adverse to the interests of their clients, employers or the public.
g. Engineers shall not accept professional employment outside of their regular work or interest without the knowledge of their employers.

© 2003 by CRC Press LLC

CANON 5. Engineers shall build their professional reputation on the merit of their services and shall not compete unfairly with others.

a. Engineers shall not give, solicit or receive either directly or indirectly, any political contribution, gratuity or unlawful consideration in order to secure work, exclusive of securing salaried positions through employment agencies.
b. Engineers should negotiate contracts for professional services fairly and on the basis of demonstrated competence and qualifications for the type of professional service required.
c. Engineers may request, propose or accept professional commissions on a contingent basis only under circumstances in which their professional judgments would not be compromised.
d. Engineers shall not falsify or permit misrepresentation of their academic or professional qualifications or experience.
e. Engineers shall give proper credit for engineering work to those to whom credit is due, and shall recognize the proprietary interests of others. Whenever possible, they shall name the person or persons who may be responsible for designs, inventions, writings or other accomplishments.
f. Engineers may advertise professional services in a way that does not contain misleading language or is in any other manner derogatory to the dignity of the profession. Examples of permissible advertising are as follows:
   - Professional cards in recognized, dignified publications, and listings in rosters or directories published by responsible organizations, provided that the cards or listings are consistent in size and content and are in a section of the publication regularly devoted to such professional cards.
   - Brochures which factually describe experience, facilities, personnel and capacity to render service, providing they are not misleading with respect to the engineer's participation in projects described.
   - Display advertising in recognized dignified business and professional publications, providing it is factual and is not misleading with respect to the engineer's extent of participation in projects described.
   - A statement of the engineers' names or the name of the firm and statement of the type of service posted on projects for which they render services.
   - Preparation or authorization of descriptive articles for the lay or technical press, which are factual and dignified. Such articles shall not imply anything more than direct participation in the project described.
   - Permission by engineers for their names to be used in commercial advertisements, such as may be published by contractors, material suppliers, etc., only by means of a modest, dignified notation acknowledging the engineers' participation in the project described. Such permission shall not include public endorsement of proprietary products.
g. Engineers shall not maliciously or falsely, directly or indirectly, injure the professional reputation, prospects, practice or employment of another engineer or indiscriminately criticize another's work.
h. Engineers shall not use equipment, supplies, laboratory or office facilities of their employers to carry on outside private practice without the consent of their employers.

CANON 6. Engineers shall act in such a manner as to uphold and enhance the honor, integrity and dignity of the engineering profession.

a. Engineers shall not knowingly act in a manner which will be derogatory to the honor, integrity or dignity of the engineering profession or knowingly engage in business or professional practices of a fraudulent, dishonest or unethical nature.

© 2003 by CRC Press LLC

CANON 7. Engineers shall continue their professional development throughout their careers, and shall provide opportunities for the professional development of those engineers under their supervision.

a. Engineers should keep current in their specialty fields by engaging in professional practice, participating in continuing education courses, reading in the technical literature and attending professional meetings and seminars.
b. Engineers should encourage their engineering employees to become registered at the earliest possible date.
c. Engineers should encourage engineering employees to attend and present papers at professional and technical society meetings.
d. Engineers shall uphold the principle of mutually satisfying relationships between employers and employees with respect to terms of employment, including professional grade descriptions, salary ranges and fringe benefits.

© 2003 by CRC Press LLC